Outline

Graph-based image processing

— Shortest Path Problem —

(Professor version)

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

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Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to each node in V.
- Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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Shortest Paths

INSTANCE A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Network

Definition

- *A* ***network*** *is a triple* $N = (E, \Gamma, \ell)$ *such that*
  - (E, Γ) *is a graph without loop; and*
  - ℓ *is a map from* $\overrightarrow{\Gamma}$ *in* $\mathbb{R}$
- *If* (E, Γ, ℓ) *is a network and if* $u \in \overrightarrow{\Gamma}$ *is an arc, the real number* $\ell(u)$ *is called the* ***length of u***

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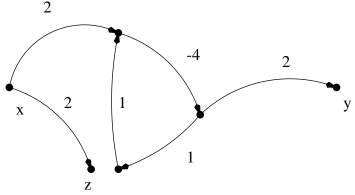

Notations

- Here, $N = (E, \Gamma, \ell)$ denotes a network, and G denotes the graph $G = (E, \Gamma)$
- If $u = (x, y)$ is an arc of G, we write $\ell(x, y)$ instead of $\ell((x, y))$

Length of a path

- Let $\pi = (x_0, \ldots, x_n)$ be a path in G
- The length of π (in N) is the sum of the length of the arcs in π:
  - $L(\pi) = \sum\{\ell(x_i, x_{i+1}) \mid 0 \leq i \leq n-1\}$

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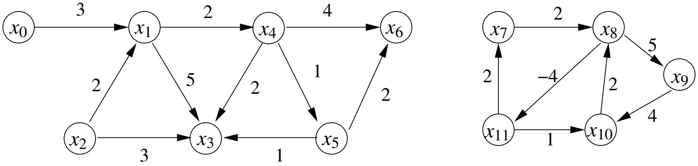
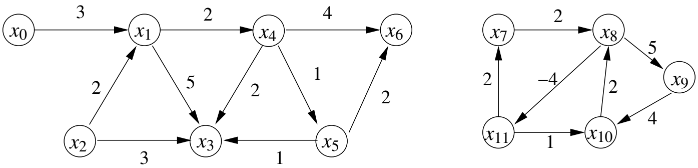


$L((x_0, x_1, x_3)) = 8$

Shortest path

- Let x and y be two vertices of G
- A **shortest path from x to y (in N)** is a path π from x to y such that the length of π is less than or equal to the length of **any other path** from x to y:
  - $\forall \pi'$ path from x to y, $L(\pi) \leq L(\pi')$

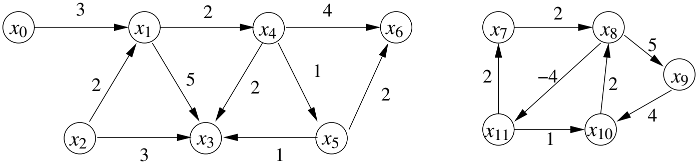
Shortest path: illustration



Example

- $\pi = (x_0, x_1, x_3)$

Shortest path: illustration



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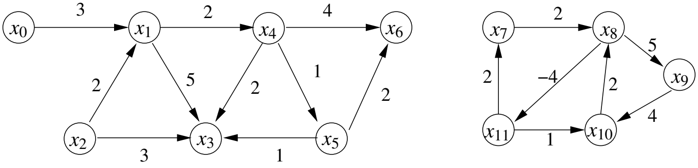

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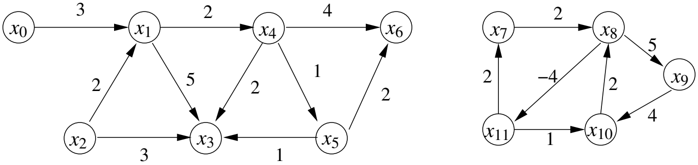

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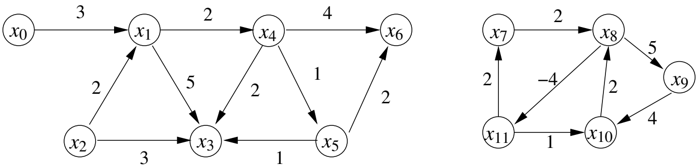
- *shortest path from* x_2 *to* x_0 *?*

Shortest path: illustration



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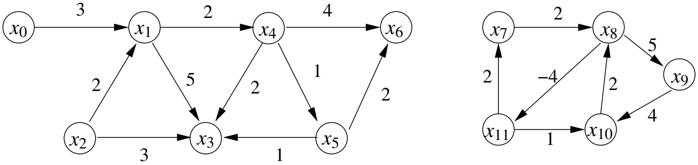

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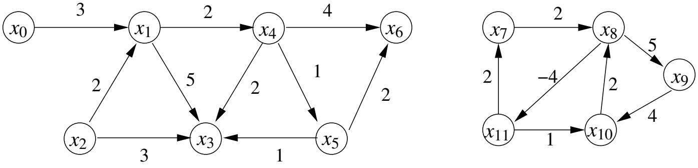
- *shortest path from* x_7 *to* x_9 *?*

Shortest path: illustration



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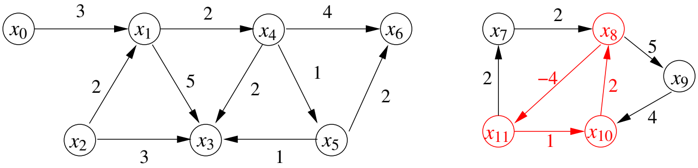
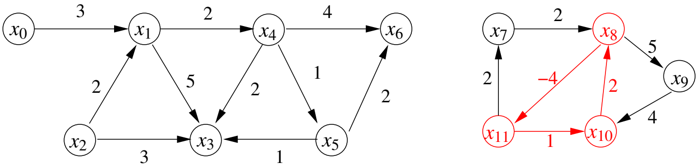
Negative circuit



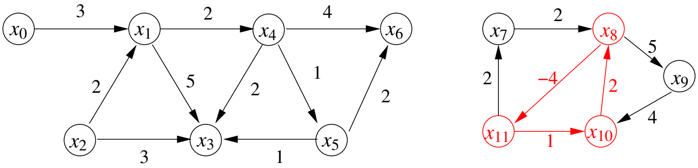
Negative circuit



Definition

- *A* ***negative circuit*** *in N is a circuit of negative length*

Negative circuit



Definition

- *A* ***negative circuit*** *in N is a circuit of negative length*

Remark. If a strongly connected component has a negative circuit, then there is no shortest path between any two arbitrary vertices of this component

Existence of a shortest path

Property

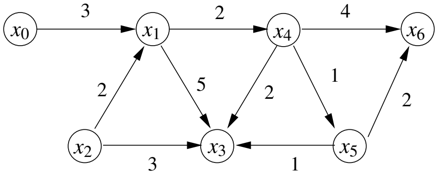
- *There exists a shortest path from x to any other vertex in E if and only if*
  - $\forall y \in E$, $\exists$ *a path from x to y*
  - *there is no negative circuit in N*

Shortest path or negative circuit?

- There exist algorithms for
  1. Finding shortest paths if they exist and
  2. Detecting if a graph has a negative circuit
- For instance, Bellman algorithm

Positive lengths network

- A **positive length network** is a network (E, Γ, ℓ) such that:
  - $\forall u \in \overrightarrow{\Gamma}$, $\ell(u) \geq 0$

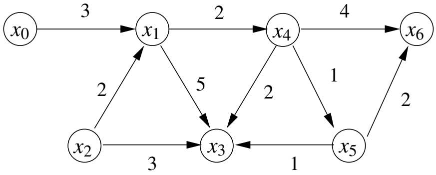


Positive lengths network

- A **positive length network** is a network (E, Γ, ℓ) such that:
  - $\forall u \in \overrightarrow{\Gamma}, \ \ell(u) \geq 0$

Property

- *If (E, Γ, ℓ) is a positive lengths network, then $\forall x, y \in E$*
  - *$\exists$ a path from x to y $\Leftrightarrow$ $\exists$ a shortest path from x à y*

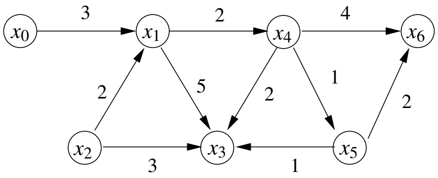


Shortest paths

- Let $N = (E, \Gamma, \ell)$ be a positive lengths network, let $x \in E$
- We define the map $L_x : E \to \mathbb{R} \cup \{\infty\}$ by:

$$L_x(y) = \begin{cases} \text{the length of a shortest path from } x \text{ to } y\text{, if such path exists} \\ \infty \text{ , otherwise} \end{cases}$$

Illustration: the map L_x



Example

$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$							

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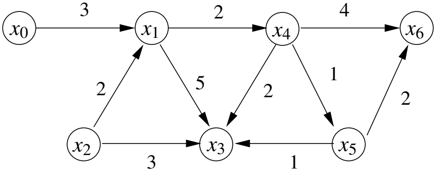

Example

$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
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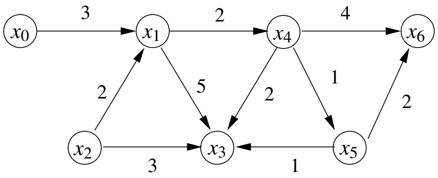

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$L_{x_0}(y) =$	0	3					

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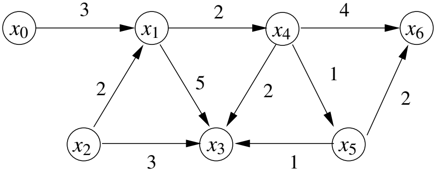

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$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$	0	3	∞				

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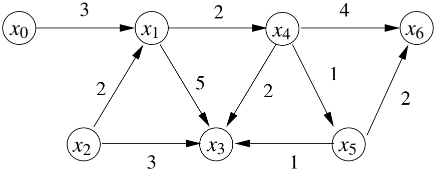

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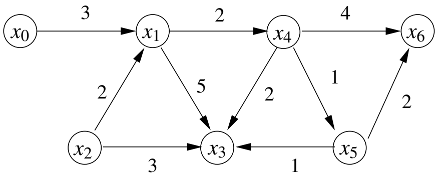

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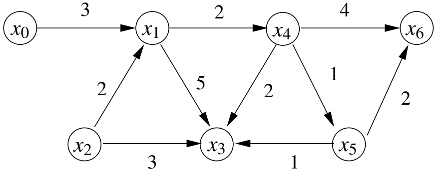

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$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
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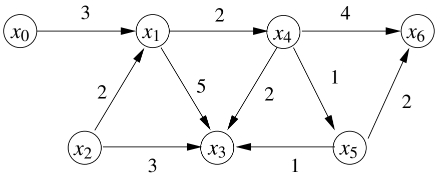

Example

$y =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6
$L_{x_0}(y) =$	0	3	∞	7	5	6	8

Graph-based image processing

— Algorithms for Single Source Shortest Path —

(Professor version)

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Problems

1. Given a network (E, Γ, ℓ) and two vertices x and y in E
   - Find a shortest path from x to y
   - Find the length $L_x(y)$ of a shortest path from x to y
2. Given a network (E, Γ, ℓ) and a vertex x in E
   - Find for each vertex y in E the length $L_x(y)$ of a shortest path from x to y
3. Given a network (E, Γ, ℓ)
   - Find, for each pair x, y of vertices in E, the length of a shortest path from x to y
4. Having solved problem 2
   - Solve problem 1

Dijkstra algorithm

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Computing the lengths of shortest paths

Algorithm DIJKSTRA (Data: (E, Γ, ℓ), $n = |E|$, $x \in E$;

Result: L_x)

$\overline{S} := \emptyset$;

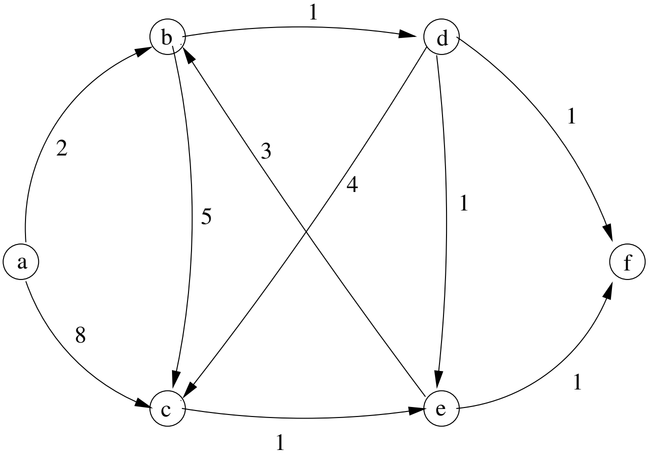
For each $y \in E$ **Do** $L_x[y] = \infty$; $\overline{S} := \overline{S} \cup \{y\}$;

$L_x[x] := 0$; $k := 0$; $\mu := 0$;

While $k < n$ and $\mu \neq \infty$ **Do**

- Extract a vertex $y^\star \in \overline{S}$ such that $L_x[y^\star] = \min\{L_x[y], y \in \overline{S}\}$
- $k++$; $\mu := L_x[y^\star]$;
- **For each** $y \in \Gamma(y^\star) \cap \overline{S}$ **Do**
    - $L_x[y] := \min\{L_x[y], L_x[y^\star] + \ell(y^\star, y)\}$;

- *Exercise.* Execute "by hand" Dijsktra algorithm on the following network with $x = a$, and on any positive length network of your choice



Loop invariant of Dijkstra algorithm (# 1)

- Let $x \in E$ and $\mu \in \mathbb{R}$
- A subset S of E is called a μ-separating (for x) if the two following conditions hold true:

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  1. S contains any vertex y such that the length $L_x(y)$ of a shortest path from x to y is less than μ
  2. $\overline{S} = E \setminus S$ contains any vertex y such that the length of a shortest path from x to y is greater than μ

Loop invariant of Dijkstra algorithm (# 2)

- Let $x \in E$, let $\mu \in \mathbb{R}$, and let S be a set that is μ-separating for x
- An S-path is a path whose intermediary vertices are all in S

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- *Then, $L_x^S(y^\star) = L_x(y^\star)$*
- *Thus, $S \cup \{y^\star\}$ is a set that is μ'-separating with $\mu' = L_x^S(y^\star)$*

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Algorithm: Shortes path algorithm – Dijkstra)

input : A graph $G = (V, E)$, a weight map W and a source node s.
output: The distances of the vertices from s

Let S be the set of explored nodes;
foreach $u \in S$ **do** store distance $d[u] = \infty$;
Initially $d[s] = 0$ and $S = s$;
while $S \neq V$ **do**
    Select a node $v \notin S$ with at least one edge from S for which $d'(v) = \min_{e=(u,v):u \in S} d[u] + W(e)$ is as small as possible;
    Add v to S and define d[v] = d'[v];
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- Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u.
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Algorithm: Shortes path algorithm – Dijkstra)

input : A graph $G = (V, E)$, a weight map W and a source node s.
output: The distances of the vertices from s

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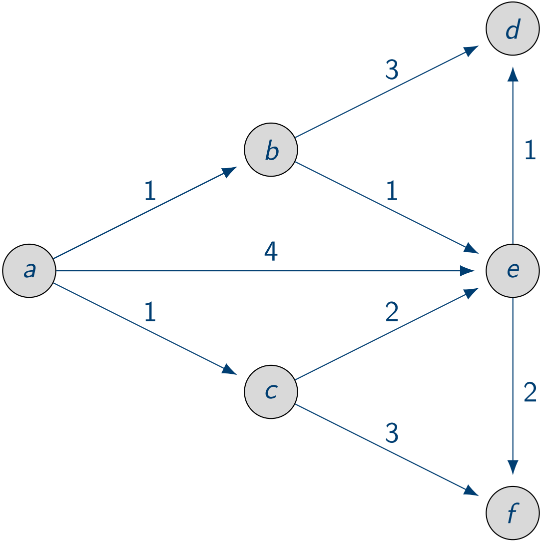
- Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimises $d'(v)$.

Example of Dijkstra's Algorithm



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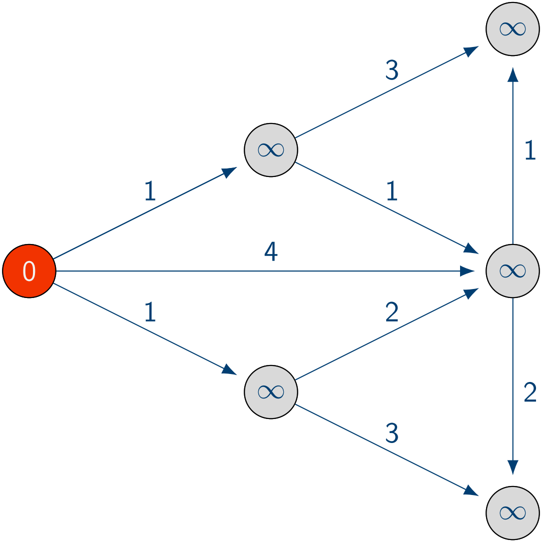

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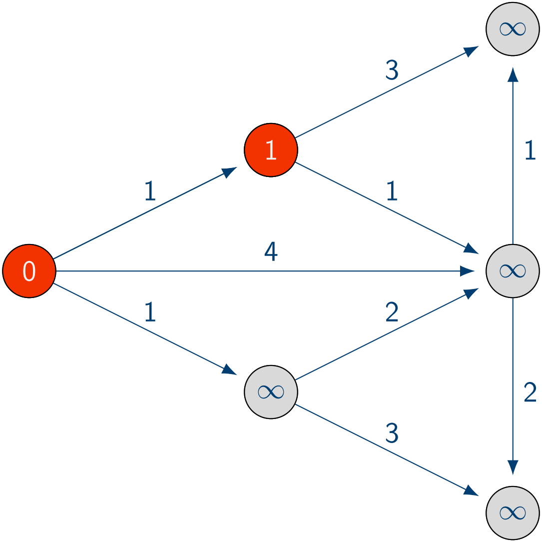

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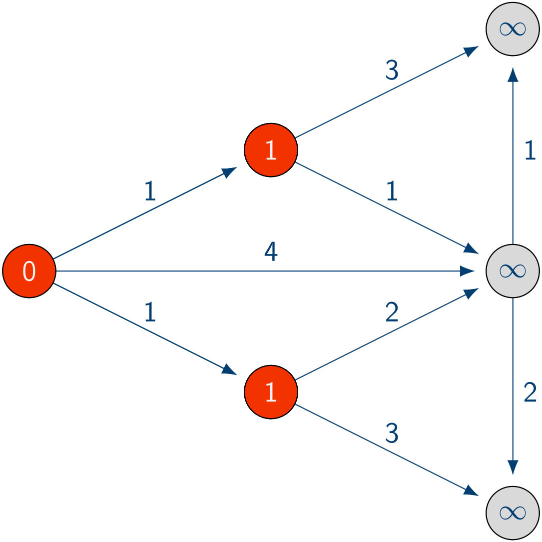

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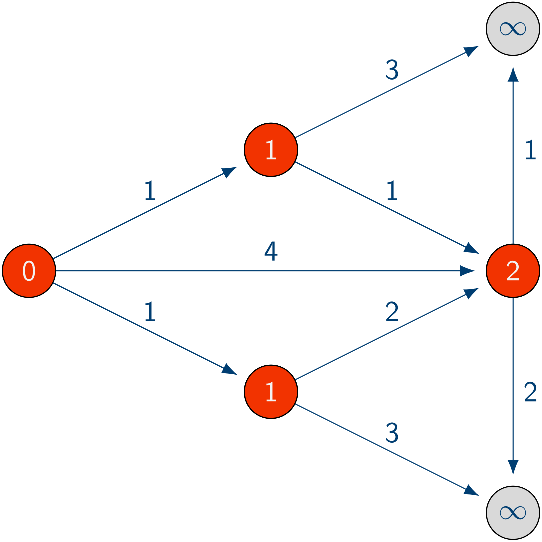

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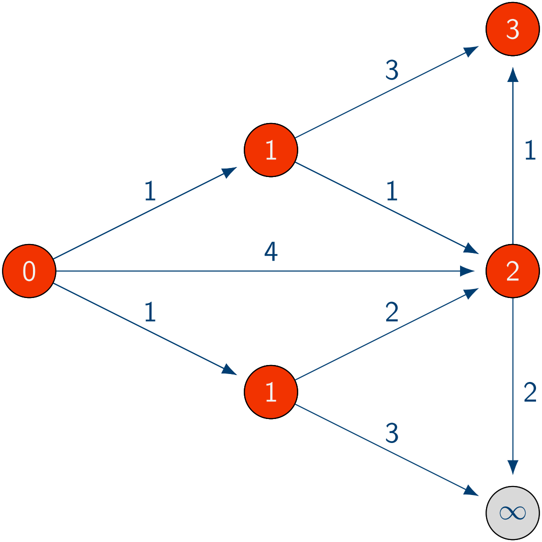

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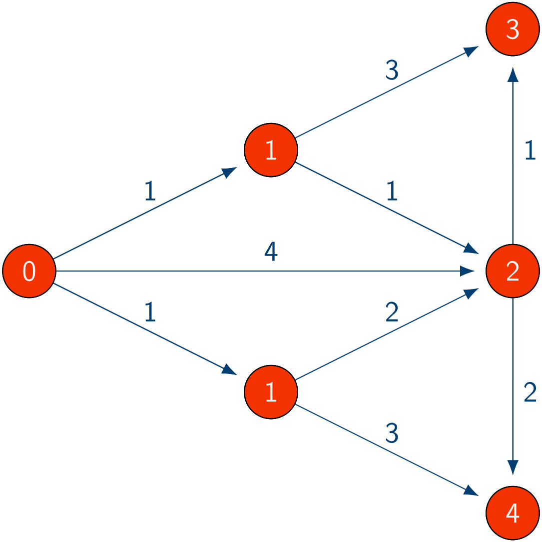

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.
  - Base case: $|S| = 1$. The only node in S is s.
  - Inductive step: we add the node v to S. Let u be the v's predecessor on the path P_v. Could there be a shorter path P from s to v?

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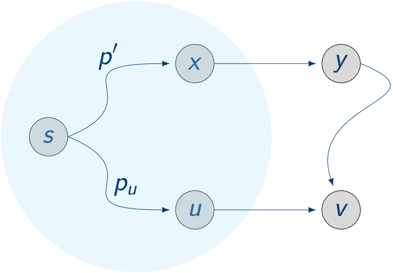

The alternate $s - v$ path P through x and y already too long by the time it had left the set S

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

Algorithm: Shortes path algorithm – Dijkstra)

input : A graph $G = (V, E)$, a weight map W and a source node s.
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- How many iterations are there of the while loop? .

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- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $v \notin S$, compute

$$\min_{e=(u,v),u \in S} d(u) + l_e.$$

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- Determine the next node v to add to S using ExtractMin.
- After adding v, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$, update w's key using ChangeKey.

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively.

Single Source Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length l_e. Note that the weights may be negative.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to all other nodes in V.
- Aside: If G is undirected, convert to a directed graph by replacing each edge in G by two directed edges.

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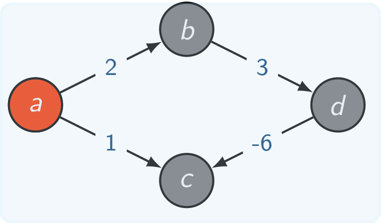
Shortest Paths

INSTANCE A directed graph $G(V, E)$, a function $l : E \to \mathbb{R}$, and a node $s \in V$

SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Bellman-Ford Algorithm

Dijkstra – Can fail if negative edge costs.



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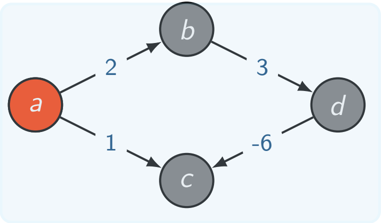
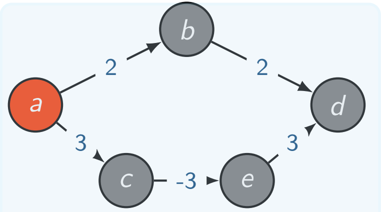


Re-weighting – Adding a constant to every edge weight can fail



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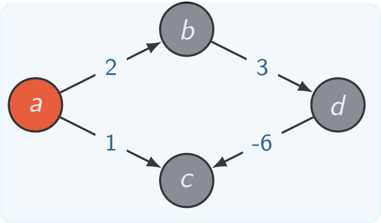

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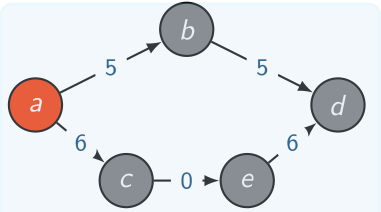
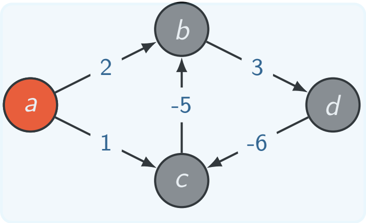


Bellman-Ford Algorithm

If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; otherwise, there exists one that is simple.



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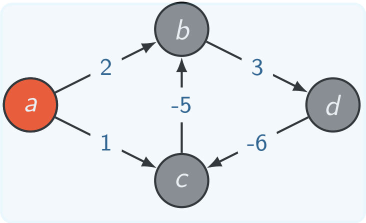

The Bellman-Ford algorithm is a way to find single source shortest paths in a graph with negative edge weights (but no negative cycles).

Bellman-Ford Algorithm

OPT(i, v) = length of shortest v-t path P using at most i edges.

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Bellman-Ford Algorithm

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- Case 1: P uses at most $i-1$ edges.

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- Case 2: P uses exactly i edges
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Bellman-Ford Algorithm

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  - if (v,w) is first edge, then OPT uses (v,w), and then selects best w-t path using at most $i-1$ edges

$$OPT(i,v) = \begin{cases} 0, & \textit{if } \; i = 0 \\ \min \begin{cases} OPT(i-1,v) \\ \min\{OPT(i-1,w) + c_{vw}\} \end{cases}, & \textit{otherwise} \end{cases}$$

A Faster implementation of Dijkstra's Algorithm

Algorithm: Shortest path algorithm – Bellman-Ford

input : A graph $G = (V, E)$, a weight map W and a source node s.
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foreach $v \in V$ **do** $d[0, u] = \infty$;
Initially $d[0, s] = 0$;

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- For finding the shortest paths, it is necessary to maintain a successor for each table entry.

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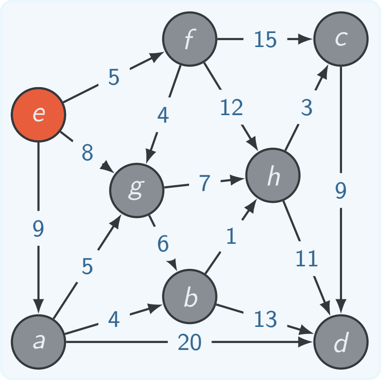
How to detect negative cycles?

Shortest path – an example



Compute the shortest path from e to all other nodes!

Complexity

Complexity

- *Initialization:* $O(n)$
- **While** *loop (line 4):* $O(n)$
- *Extract (line 5):* $O(n^2)$
- **For each** *loop (line 7):* $O(n + m)$
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- *DIJKSTRA:* $O(n)$
- *can be easily reduced to* $O(n \log(n) + m)$

Exercise

- Propose an algorithm whose **data** are:
  - a positive lengths network N
  - a pair (x, y) of vertices
- and whose **result** is:
  - a shortest path from x to y if such path exists

Help. Start by computing the lengths $L_x(z)$ for all vertices $z \in E$ using Dijkstra algorithm.

Acknowledgement

Thanks to the Prof. Jean Cousty at ESIEE/France that gently sent me the slides used in the Morpho, Graph and Image course. Some slides of the Graph-based Image Processing course at PPGINF/PUC Minas under supervision of Prof. Silvio Guimarães will be adapted versions of that course.

- Course - MorphoGraph and Imagery
  https://perso.esiee.fr/ coustyj/EnglishMorphoGraph/
- Jean Cousty
  - ESIEE Paris, Département Informatique
  - Université Paris-Est, LIGM (UMR CNRS, ESIEE...)
  - E-mail: `j.cousty@esiee.fr`