Programa de Pós-graduação em INFORMÁTICA

PUC Minas

IM SCIENCE

Graph-based image processing

— Trees —

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Aug 2022

Outline of the lecture

Graph-based image processing

— Trees and arborescences —

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Preliminary remark

- In the sequel, the symbol G denotes an asymmetric graph (E, Γ)

Isthmus

Definition

- *Let* $u \in \overrightarrow{\Gamma}$
- *The arc u is an* isthmus for G *if the number of connected components of* $(E, \overrightarrow{\Gamma} \setminus \{u\})$ *is greater than the one of G*

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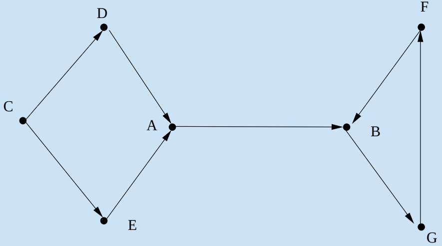
Example



Exercise. *Give all isthmus of this graph*

Isthmus and cycles

Property

- *The arc u is an isthmus for G if and only if the arc u does not appear in any cycle of G*

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- *If G is connected, then $m \geq n - 1$*

Isthmus and cycles

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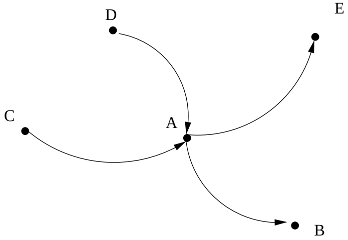
Property

- *Let $n = |E|$ and $m = |\overrightarrow{\Gamma}|$*
- *If G is connected, then $m \geq n - 1$*
- *If G has no cycle, then $m \leq n - 1$*

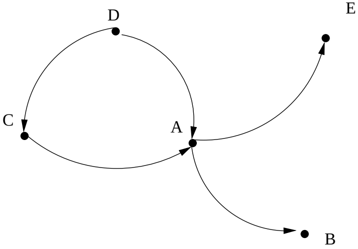
Tree

Definition

- *A* tree *is a connected graph that has no cycle*



A tree



A graph that is not a tree

Tree

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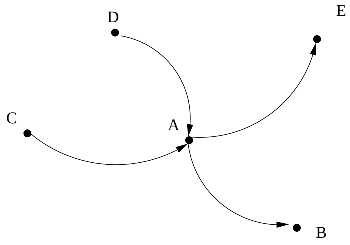
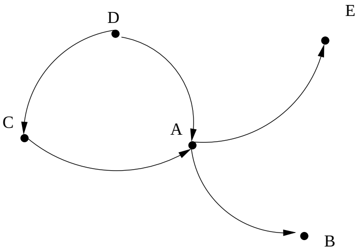
- *A* tree *is a connected graph that has no cycle*



A tree



A graph that is not a tree

Remark. In a tree, each arc is an isthmus

Tree: characterization

Property

- *The four following statements are equivalent:*
  1. G *is a tree*
  2. G *is connected and* $m = n - 1$
  3. G *has no cycle and* $m = n - 1$
  4. *for any pair of two distinct vertices* x *and* y *in* E*, there exists a unique elementary undirected path from* x *to* y

Root

Definition

- *Let x be a vertex in E*
- *x is a* root of G *if E is the transitive closure of $\{x\}$ (i.e., if, for each vertex $y \in E$, there exists a path from x to y)*

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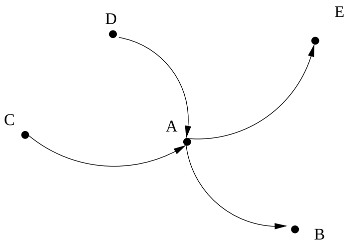

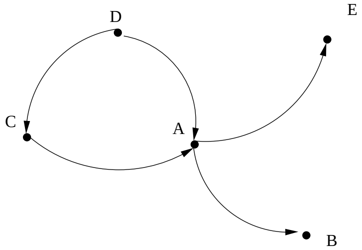

Exercise. Give the roots of these graphs

Arborescence

Definition

- *An* arborescence *is a tree for which there is a root*

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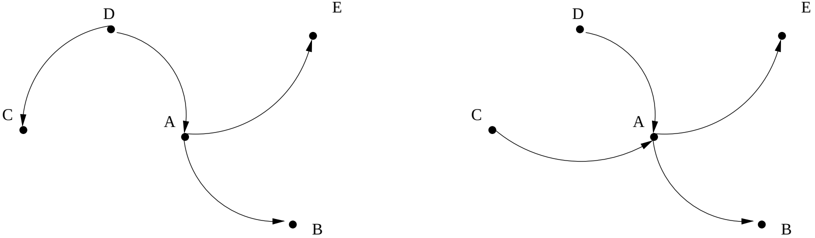


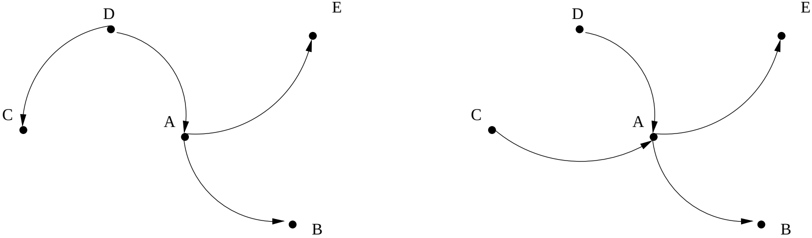
An arborescence (left)

A tree that is not an arborescence (right)

Arborescence

Definition

- *An* arborescence *is a tree for which there is a root*
- *A vertex x of an arborescence is a* leaf *if it has no successor*

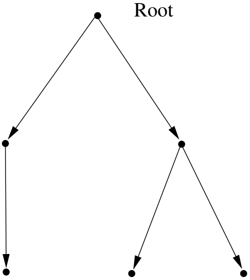


An arborescence (left)
A tree that is not an arborescence (right)

Properties of arborescences

Property

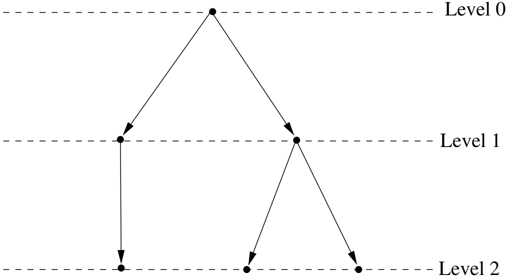
- *If G is an arborescence, there exists a unique vertex in E that is a root of G*



Properties of arborescences

Property

- *If G is an arborescence, there exists a unique vertex in E that is a root of G*
- *An arborescence can be decomposed into levels*



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Graph-based image processing

— Examples of application —

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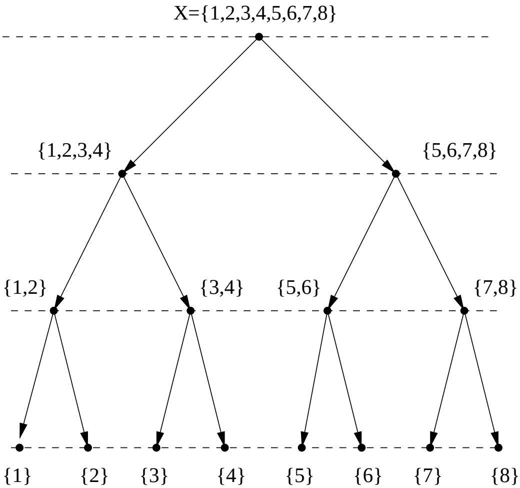
Problem 1: counterfeit coins

- Let $P = \{p_1, \ldots, p_8\}$ be a set of 8 coins
  - 7 coins have the same weight
  - 1 is lighter than the others
- Let us consider a balance with two states $(<, \geq)$
- *Problem.* Find the lightest coin

Decision arborescence: example

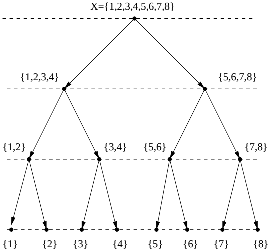


Decision arborescence for the counterfeit coins problem

Decision arborescence

Definition

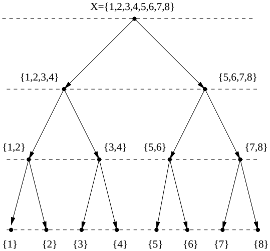
- *Let P be a finite set (of possible solutions)*



Decision arborescence

Definition

- *Let P be a finite set (of possible solutions)*
- *A* **decision arborescence for P** *is a pair (G, f) such that*
  - $G = (E, \Gamma)$ *is an arborescence of root r*
  - f *is a map from E into $\mathcal{P}(P)$ that satisfies*
    - $f(r) = P$
    - $\forall x \in E \mid \Gamma(x) \neq \emptyset$, $f(x) = \cup\{f(y) \mid y \in \Gamma(x)\}$
    - $\forall x \in E \mid \Gamma(x) \neq \emptyset$, $|f(x)| = 1$



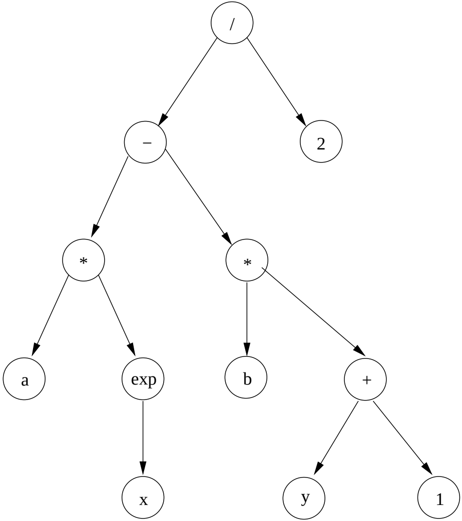
- Same as 1 but the considered balance has 3 states $(<, =, >)$ (instead of 2)

Exercise.

- Give a decision arborescence for this problem
- Give the number of necessary weighing to find the counterfeit coin?

Arithmetic expression evaluation

- $(A) = \frac{ae^x - b(y+1)}{2}$



Ordered arborescence

Definition

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- *Any ordered graph $(E, \mathring{\Gamma})$ induces a graph (E, Γ)*

Ordered arborescence

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- *Thus, $\mathring{\Gamma}(x) = (y_0, \ldots, y_\ell)$ for a given value ℓ*
- *Any ordered graph $(E, \mathring{\Gamma})$ induces a graph (E, Γ)*
- *$(E, \mathring{\Gamma})$ is an* ordered arborescence *if the graph (E, Γ) induced by $(E, \mathring{\Gamma})$ is an arborescence*

A search arborescence

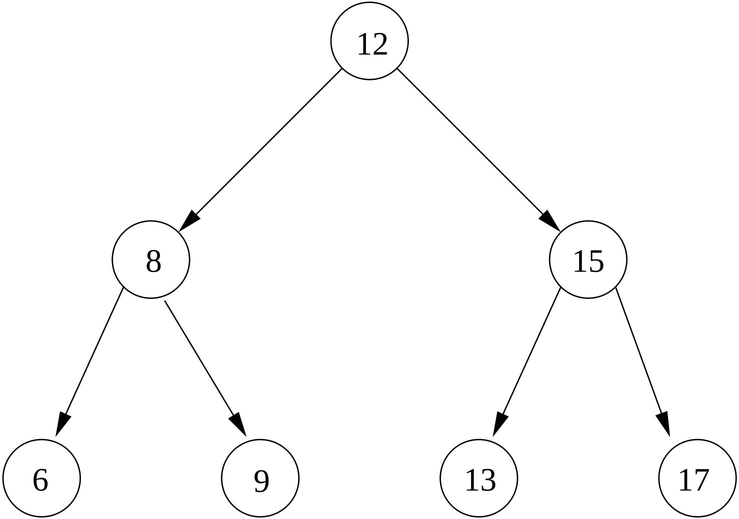
A search arborescence

- Let $\mathbb{V}$ be a set totally ordered by the relation $\leq$
- Let $X \subseteq \mathbb{V}$

A search arborescence

- Let $\mathbb{V}$ be a set totally ordered by the relation $\leq$
- Let $X \subseteq \mathbb{V}$
- A *search arborescence on X* is an ordered arborescence $(X, \mathring{\Gamma})$ such that
  - $\forall x \in X$, $0 \leq |\Gamma(x)| \leq 2$
  - $\forall x \in X \mid \mathring{\Gamma}(x) = (x_1, x_2)$, any element in the sub-arborescence rooted in x_1 is smaller than x and any element in the sub-arborescence rooted in x_2 is greater than x

Example



Search arborescence for the relation $\leq$ defined on $\mathbb{V} = \mathbb{N}$, with $X = \{6, 8, 9, 12, 15, 13, 17\}$.

Graph-based image processing

— Exercises —

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Exercises

- Write an algorithm that tests if a given value ν belongs to a search arborescence $(X, \mathring{\Gamma})$
- Write an algorithm that prints the value of an arborescence $(X, \mathring{\Gamma})$ in increasing order

Acknowledgement

Thanks to the Prof. Jean Cousty at ESIEE/France that gently sent me the slides used in the Morpho, Graph and Image course. Some slides of the Graph-based Image Processing course at PPGINF/PUC Minas under supervision of Prof. Silvio Guimarães will be adapted versions of that course.

- Course - MorphoGraph and Imagery
  https://perso.esiee.fr/ coustyj/EnglishMorphoGraph/
- Jean Cousty
  - ESIEE Paris, Département Informatique
  - Université Paris-Est, LIGM (UMR CNRS, ESIEE...)
  - E-mail: `j.cousty@esiee.fr`