Graph-based image processing

— Connected operators —

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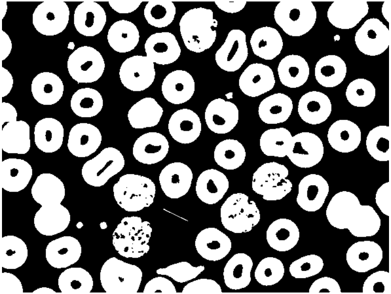
Aug 2022

Introduction: a problem seen during practical session



- Problem: Fill the holes of the cells

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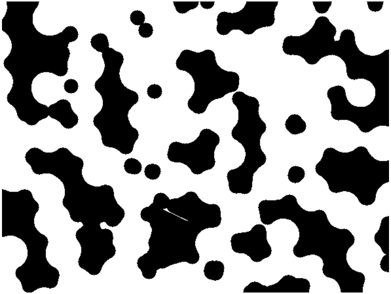
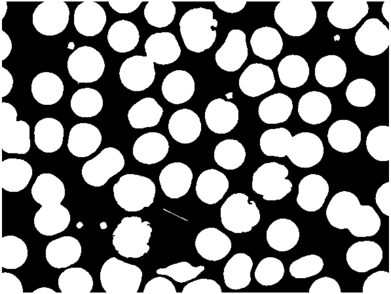


- Problem: Fill the holes of the cells
- Closing by a structuring element,
  - Holes are closed
  - But contours "have moved"

Introduction: a problem seen during practical session



- Problem: Fill the holes of the cells
- Closing by a structuring element,
  - Holes are closed
  - But contours "have moved"
- A connected closing by reconstruction

Outline of the lecture

1. Connectivity: reminder

2. Connected openings/closings: the case of sets

3. Connected opening: the case of grayscale images

4. Component tree

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— Connectivity: reminder —

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Connexity

- E is a set
- $G = (E, \Gamma)$ is a graph

Connected components of a set

Definition

- *Let* $X \subseteq E$
- *The* ***subgraph of G induced by X*** *is the graph* (X, Γ_X) *such that*
  - $\overrightarrow{\Gamma_X} = \{(x, y) \in \overrightarrow{\Gamma} \mid x \in X, y \in X\}$

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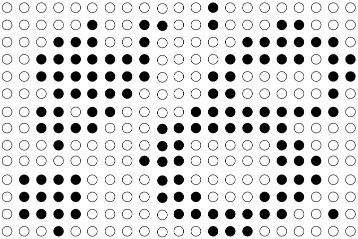
- *A connected component of the subgraph of G induced by X is simply called a **connected component of X***



$X \subseteq E$ (in black)

$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

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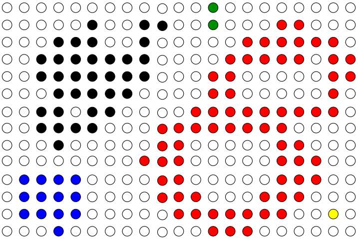

Partition of X into connected components

$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

Connected components of a set

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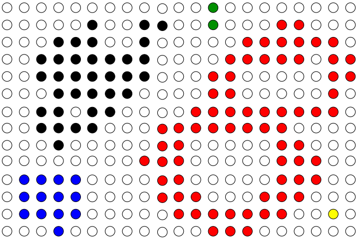
- *A connected component of the subgraph of G induced by X is simply called a* ***connected component of X***
- *$\mathcal{C}_X$ denotes the set of all connected components of X*



$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

Graph-based image processing

— Connected openings/closings: the case of sets —

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Connected openings/closings

Definition

- *A filter γ on E is a* ***connected opening*** *if*
  - $\forall X \in \mathcal{P}(E), \mathcal{C}_{\gamma(X)} \subseteq \mathcal{C}_X$
- *A filter ϕ on E is a* ***connected closing*** *if*
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Property

- *A connected opening γ is an opening*
  - $\forall X, Y \in \mathcal{P}(E), X \subseteq Y \implies \gamma(X) \subseteq \gamma(Y)$ *(increasing)*
  - $\forall X \in \mathcal{P}(E), \gamma(\gamma(X)) = \gamma(X)$ *(idempotent)*
  - $\forall X \in \mathcal{P}(E), \gamma(X) \subseteq X$ *(anti-extensive)*

Connected openings/closings

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  - $\forall X \in \mathcal{P}(E), X \subseteq \gamma(X)$ *(extensive)*

Example: area connected opening

Definition

- *Let* $\lambda \in \mathbb{N}$
- *The* ***area connected opening of parameter*** λ *is the operator* α_λ *on* E *defined by*
  - $\forall X \in \mathcal{P}(E)$, $\alpha_\lambda(X) = \cup\{C \in \mathcal{C}_X \mid |C| \geq \lambda\}$

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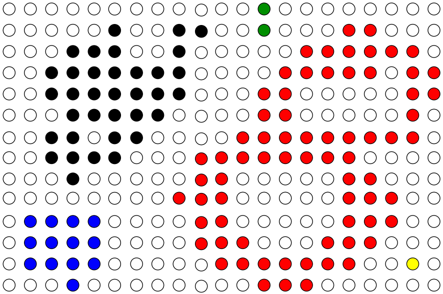

$X \subseteq E$

$(E = \mathbb{Z}^2 \text{ et } \Gamma = \Gamma_4)$

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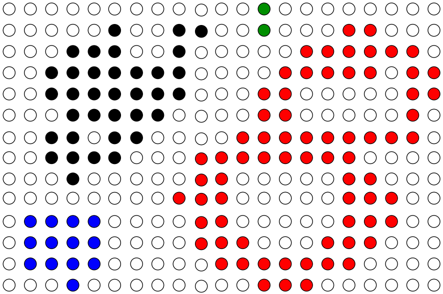

$\alpha_1(X)$

$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

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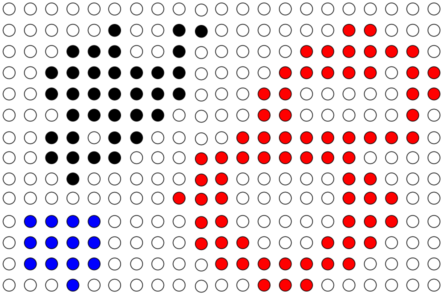

$\alpha_2(X)$

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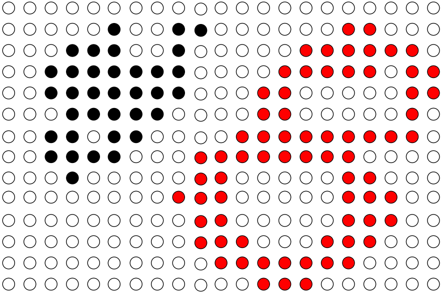

$\alpha_{13}(X)$

$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

Example: connected opening by reconstruction

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- *Let* $M \subseteq E$
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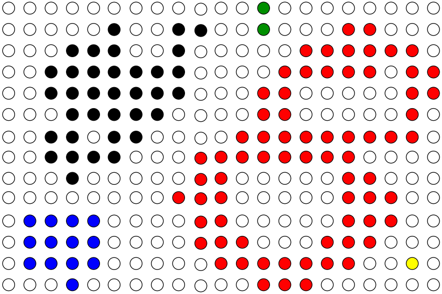

$X \subseteq E$

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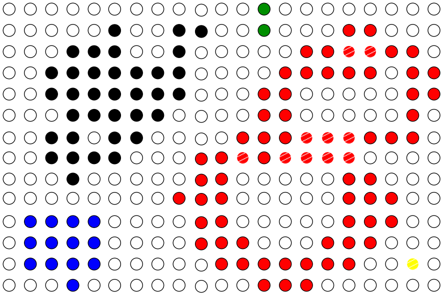

$X \subseteq E$ et M (hatched dots)

$(E = \mathbb{Z}^2$ et $\Gamma = \Gamma_4)$

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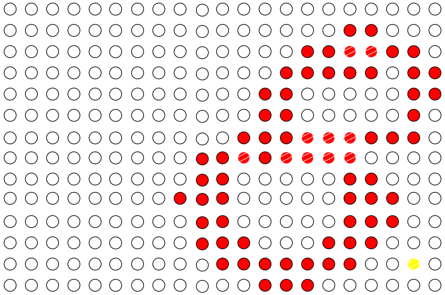

$\psi_M(X)$

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Properties of connected filters

Property

- γ *is a connected opening* $\Leftrightarrow \star\gamma$ *is a connected closing*

Properties of connected filters

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- γ *is a connected opening* $\Leftrightarrow \star\gamma$ *is a connected closing*

Property (contour preservation)

- *If* γ *is a connected opening or closing, then*
  - $\forall X \in \mathcal{P}(E), \forall (x,y) \in \overrightarrow{\Gamma},$
    $|\{x,y\} \cap \gamma(X)| = 1 \implies |\{x,y\} \cap X| = 1$

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— Connected opening: the case of grayscale images —

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Grayscale connected opening/closing

Definition (connected opening)

- *An operator γ on $\mathcal{I}$ is a* **connected opening (on $\mathcal{I}$)** *if*
  - $\forall F \in \mathcal{I}$, $\gamma(\gamma(F)) = \gamma(F)$ *(idempotence)*
  - $\forall F, H \in \mathcal{I}, \forall k \in \mathbb{Z}$, $F_k \subseteq H_k \implies \gamma(F)_k \subseteq \gamma(H)_k$ *(increasing)*
  - $\forall F \in \mathcal{I}$, $\forall k \in \mathbb{Z}$, $\mathcal{C}_{[\gamma(F)_k]} \subseteq \mathcal{C}_{[F_k]}$ *(connected, anti-extensive)*

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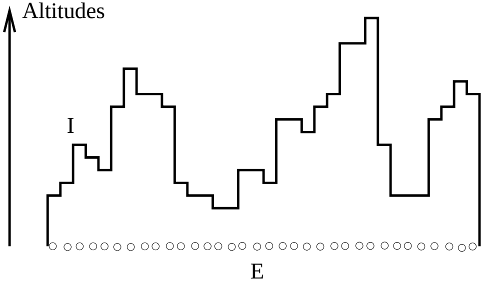
Examples

- The stack operator induced by α_λ (where $\lambda \in \mathbb{N}$) is a connected opening on $\mathcal{I}$



$$E = \mathbb{Z},\ \forall x \in E, \Gamma(x) = \{x, x-1, x+1\}$$

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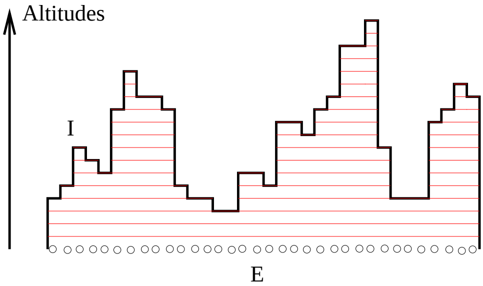

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Connected components of the level sets (red)

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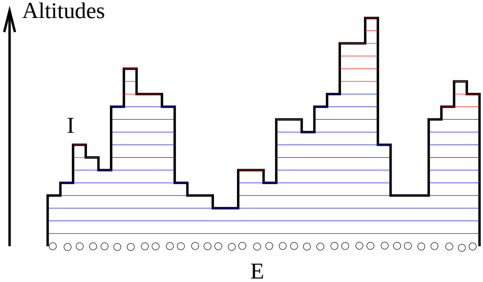

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Connected components of area greater than 3 (blue)

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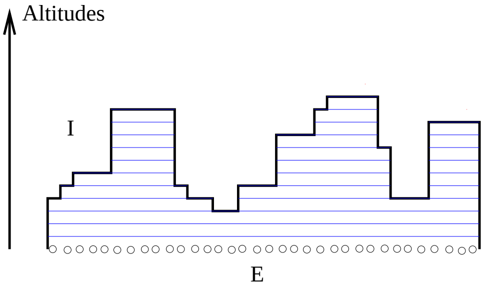

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$$\alpha_3(I)$$

Examples

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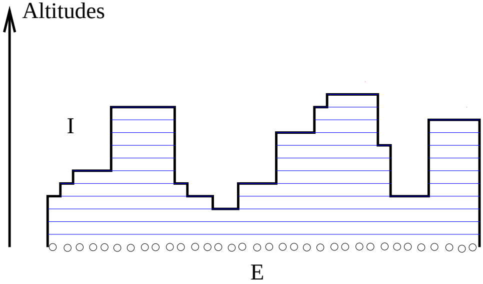
- The stack operator induced by ψ_M (where $M \subseteq E$) is a connected opening on $\mathcal{I}$

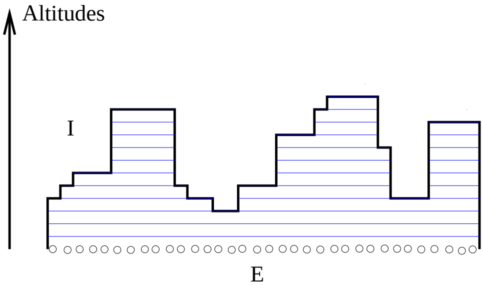


$$E = \mathbb{Z},\ \forall x \in E, \Gamma(x) = \{x, x-1, x+1\}$$

$$\alpha_3(I)$$

Examples

- The stack operator induced by α_λ (where $\lambda \in \mathbb{N}$) is a connected opening on $\mathcal{I}$
- The stack operator induced by ψ_M (where $M \subseteq E$) is a connected opening on $\mathcal{I}$
- The stack operator induced by any connected opening on E is a connected opening on $\mathcal{I}$



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$$\alpha_3(I)$$

Grayscale connected opening

Definition

- *Let* $F \in \mathcal{I}$
- *The* ***opening by reconstruction of (the marker) F*** *is the operator* ψ_F *on* $\mathcal{I}$ *defined by*
  - $\forall H \in \mathcal{I}, \forall k \in \mathbb{Z}, [\psi_F(H)]_k = \psi_{[F_k]}(H_k)$

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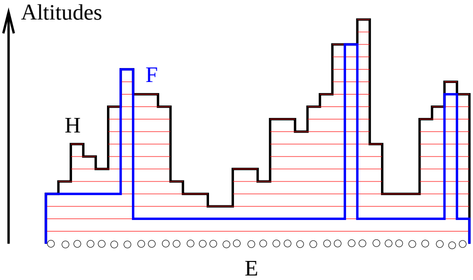

H in black, F in blue

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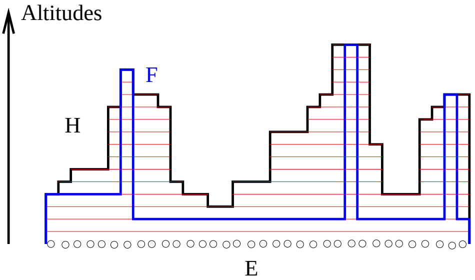

$\psi_F(H)$ (in black)

Properties

Property (duality)

- *An operator ϕ is a connected closing on $\mathcal{I}$ if and only if its dual $\star\phi$ is a connected opening on $\mathcal{I}$, where*
  - $\forall F \in \mathcal{I}, \star\phi(F) = -\phi(-F)$

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Component tree

Problem

- *How to implement a connected operator?*
  - *Data structure: component tree*
- *How to build other connected operators?*
  - *Criterion on the nodes of the component tree*

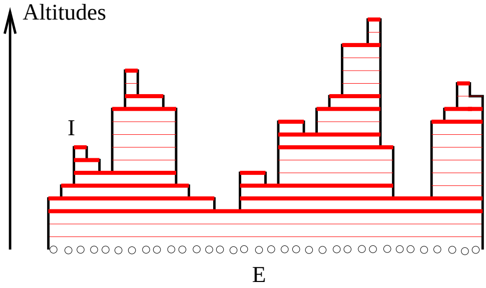
Notation

- In the following, the symbol I denotes a grayscale image on E: $I \in \mathcal{I}$

Connected component of an image

Definition

- *We denote by $\mathcal{C}_I$ the union of the sets of connected components of the level sets of I*
  - $\mathcal{C}_I = \cup\{\mathcal{C}_{[I_k]} \mid k \in \mathbb{Z}\}$
- *Each element in $\mathcal{C}_I$ is called a* connected component of I

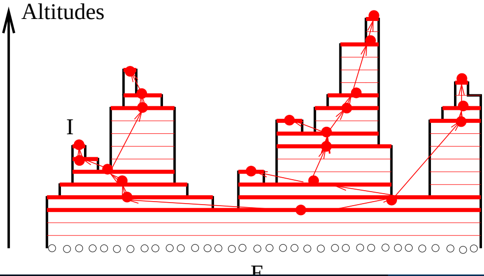


Components of I (bold red lines)

Component tree

Definition

- *Let C_1 and C_2 be two components of I in $\mathcal{C}_I$*
- *We say that **C_1 is a child of C_2 (for I)** if*
  - $C_1 \subseteq C_2$
  - $\forall C' \in \mathcal{C}_I$, *if* $C_1 \subseteq C' \subseteq C_2$ *then* $C' = C_2$ *or* $C' = C_1$



The relation "is child of" is represented by red arrows

Component tree

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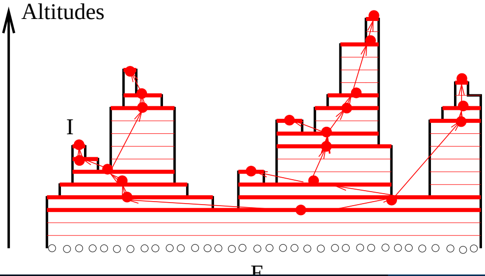
  - $\forall C' \in \mathcal{C}_I$, *if* $C_1 \subseteq C' \subseteq C_2$ *then* $C' = C_2$ *or* $C' = C_1$
- *The **component tree of I** is the graph $A_I = (\mathcal{C}_I, \Gamma_I)$ such that*
  - $(C_1, C_2) \in \Gamma_I$ *if C_2 is a child of C_1*



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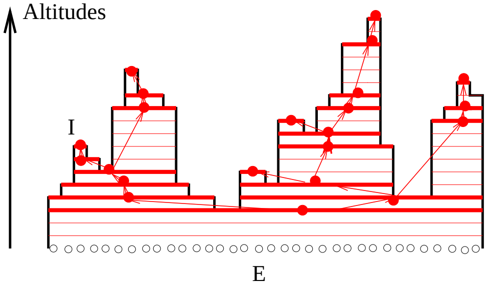
Arborescence

Property

- *If the graph (E, Γ) is connected, then the component tree of I is an arborescence of root E*

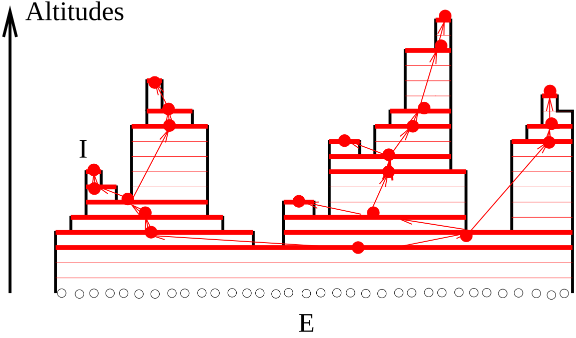


Simplifying an image by razings

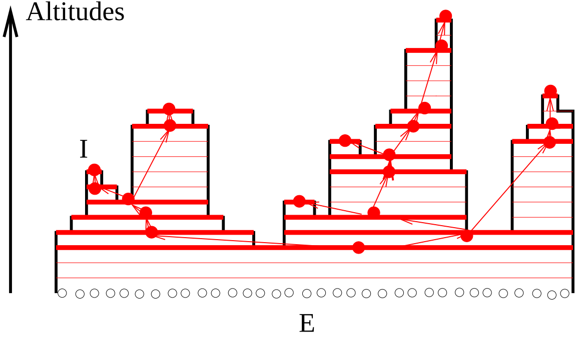
- A leaf of the component tree A_I of I is called
  - a regional maximum of I
- A leaf of the component tree A_{-I} of $-I$ is called
  - a regional minimum of I

- Given a criterion, it is possible to iteratively remove the leaves of the tree A_I
- Such removal is called a razing

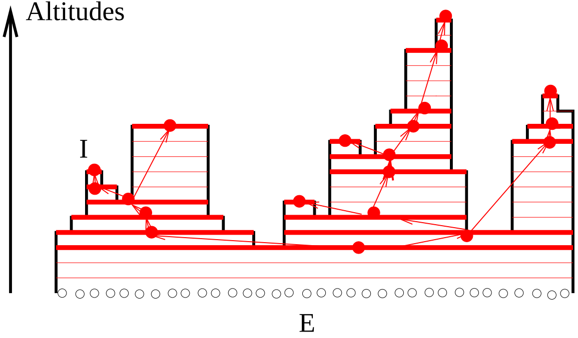
Razing: intuition



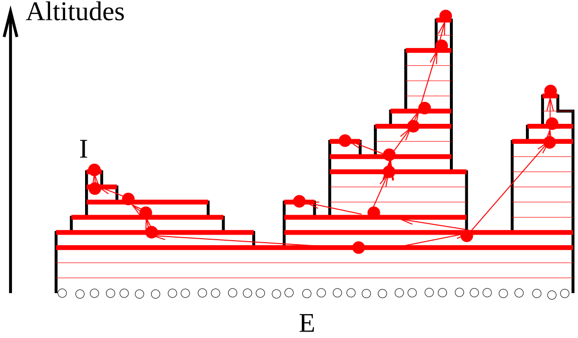
Successive razing steps



Successive razing steps



Successive razing steps



Successive razing steps
A maximum has been removed

Razing: definition

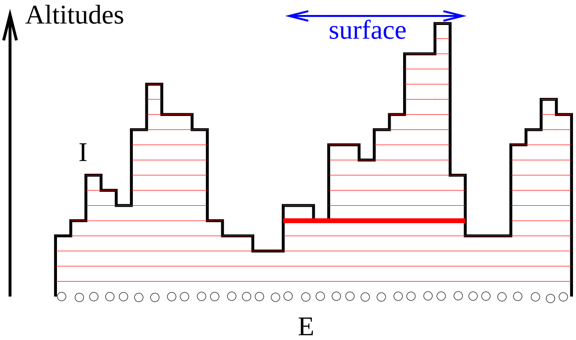
Definition

- *Let $x \in E$*
- *The* *elementary razing ν_x of I at x* *is defined b*
  - $[\nu_x(I)](y) = I(y) - 1$ *if x and y belong to a same maximum of I*
  - $[\nu_x(I)](y) = I(y)$ *otherwise*
- *An image obtained from I by composition of elementary razings is called a* *razing of I*

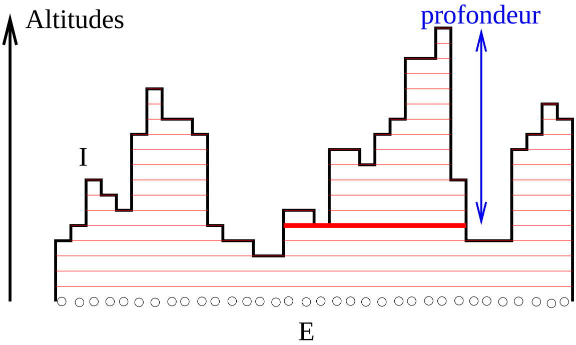
Simplifying an image by razings

- What criterion (attribute) for selecting a maxima to raze?

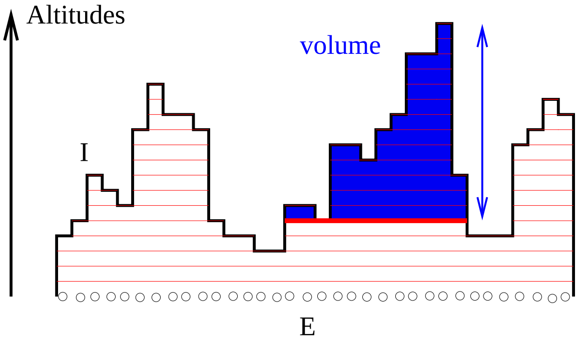
Razing by area



Razing by depth, or dynamics



Razing by volume



Flooding

- The dual of a razing is called a flooding
- A flooding removes minima by iteratively increasing their values

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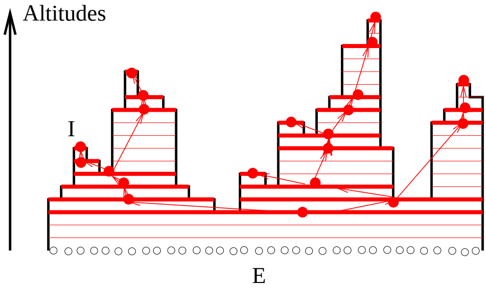
Remark. An operator produces razings is not necessarily an opening (counter-examples: razing by dynamics or volume)

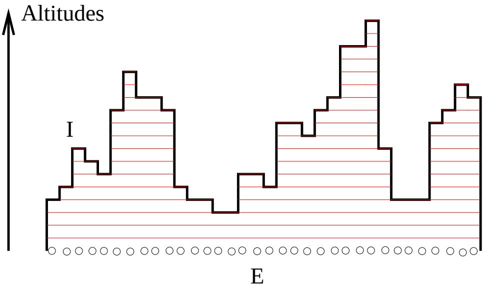
Exercise

- Draw the image F obtained by razings all components of depth is less than 12
- Compare the number of maxima of F and of I



Exercise

- Draw the image H obtained by the dual of the previous one, (*i.e.*, the flooding that removes all components of depth is less than 12)
- Help
    1. Draw $-I$
    2. Apply a razing to $-I$
    3. Inverse the obtained result
- Compare the numbers of minima of H and of I



Acknowledgement

Thanks to the Prof. Jean Cousty at ESIEE/France that gently sent me the slides used in the Morpho, Graph and Image course. Some slides of the Graph-based Image Processing course at PPGINF/PUC Minas under supervision of Prof. Silvio Guimarães will be adapted versions of that course.

- Course - MorphoGraph and Imagery
  https://perso.esiee.fr/ coustyj/EnglishMorphoGraph/
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