Mathematical Morphology and its Applications to Signal Processing

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Plan of the Course

I) Basic concepts

II) Erosions et dilations:
- a) Sets
- b) Functions
- c) Gradients

III) Openings and Closings:
- a) Morphological type
- b) Algebraic type
- c) Granulometries
- d) Top-hats

IV) Morphological filtering:
- a) Alternating Filters
- b) Sequential Alternating Filters
- c) Activity, Centre et Contrast

V) Geodesy et connectivity:
- a) Metrics et dilation
- b) Reconstruction and connectivity
- c) Numerical Geodesy

VI) Applications of Geodesy :
- a) Binary
- b) Numerical

VII) Skeletons :
- a) Ultimate Erosion
- b) Skeleton
- c) Conditional Bisector

VIII) Thinnings and Thickenings:
- a) Hit-or-Miss Transformation
- c) Thinnings and Homotopy

IX) Basins and Watersheds:
- a) SKIZ
- b) Minima and Basins
- c) Watersheds by Flooding

X) Segmentation:
- a) MISP
- b) Mosaic image and Pyramids

- I -

Basic Notions

Image Proccessing (II)

2) Extraction of Characteristics:

The aim here is to improve image quality or to exhibit some of its features. This includes in particular measurements, noise reduction, and filtering.

3) Segmentation:

Segmentation consists in partitioning the images into zones which are homogeneous according to a given criterion.

Basic structure

Linear signal processing:

The basic structure in linear signal processing is the ***vector space*** *i.e.:*

1) A set of vectors V
   and a set of scalars K such that

2) V is a <u>commutative group ;</u>
   K is a <u>field ;</u>
   There exists an external law of <u>multiplication</u> between scalars and vectors.

Mathematical morphology:

The basic structure is a ***complete lattice*** *i.e.* a set $\mathcal{L}$ such that:

1) $\mathcal{L}$ is provided with a partial ordering, *i.e.* a relation ≤ such that

$$A \leq A$$
$$A \leq B,\ B \leq A \ \Rightarrow\ A = B$$
$$A \leq B,\ B \leq C \ \Rightarrow\ A \leq C$$

2) For each family of elements $\{X_i\} \in$ P, there exists in $\mathcal{L}$:

   a Max. lower bound **,**$\{X_i\}$, called ***inf***
   a Min. upper bound **f**$\{X_i\}$, called ***sup***

Foundations of mathematical morphology

Linear signal processing:

=> The working structure is a vectorial space, and the fundamental laws are the **addition** and the **scalar product**:

- The useful operations are those preserving the working structure and commuting with the laws:

$$\varphi\left(\sum_i a_i X_i\right) = \sum_i a_i \varphi(X_i)$$

The resulting operation is the **convolution**

Mathematical morphology:

=> The working structure is the lattice, where the basic laws are the **supremum** and the **infimum**:

- The useful operations are those preserving the structure (order) and commuting with the laws:

Preserve the order:

$$\mathbf{X} \leq \mathbf{Y} \Rightarrow \varphi(\mathbf{X}) \leq \varphi(\mathbf{Y})$$

=> **Increasing operations**

Commute with Sup.:

$$\varphi\left(\bigvee_i \mathbf{X}_i\right) = \bigvee_i \varphi(\mathbf{X}_i)$$

=> **Dilation**

Commute with Inf.:

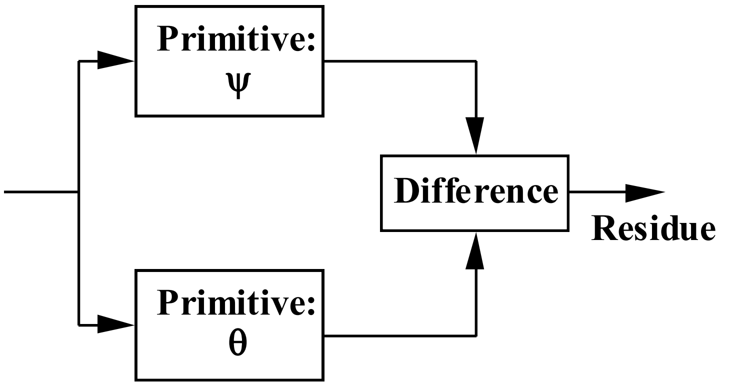
$$\varphi\left(\bigwedge_i \mathbf{X}_i\right) = \bigwedge_i \varphi(\mathbf{X}_i)$$

=> **Erosion**

Notion of residues in morphology

- The theory of morphological filters has highlighted the increasing and idempotence properties, as well as the ordering rules between transformations.

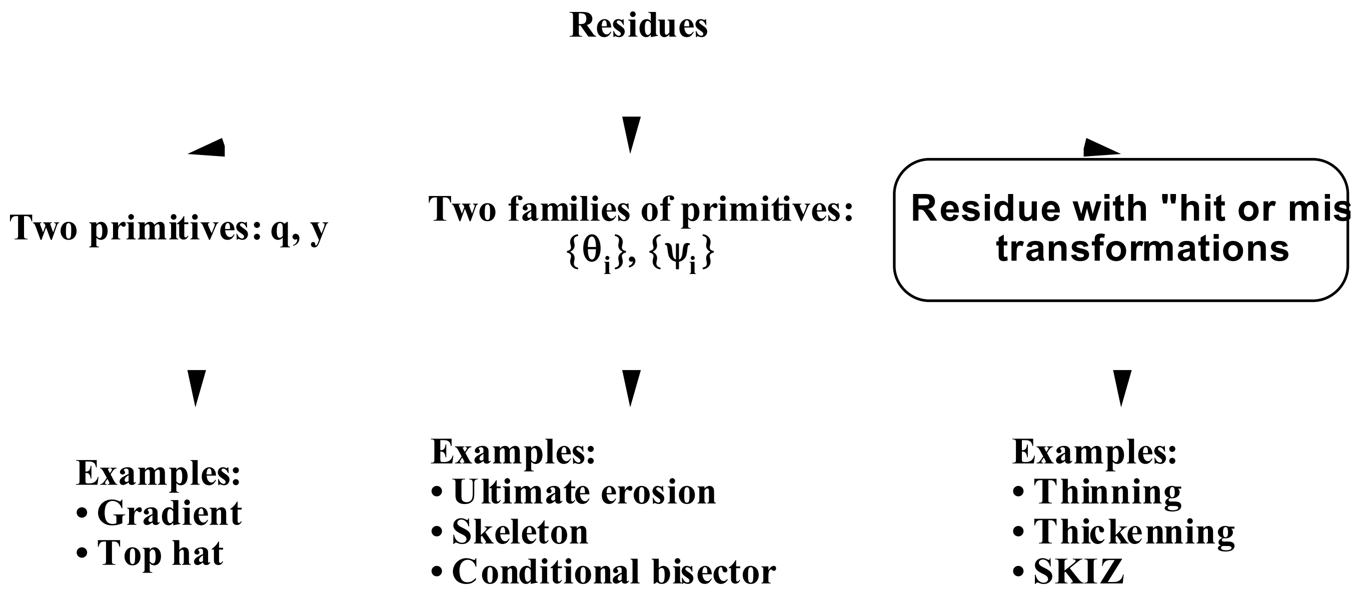
- There is a family of transformations which studies the difference between two (or many) basic transformations. Their common basis relies on the notion of **difference** also called **residue**.



Classification of residues

- The residues that are used in practice can be classified in three groups:

1) Residues of two primitives
2) Residues of two family of primitives
3) Residues relying on "hit or miss" transformations



Section II : Dilation, Erosion

Basic Notions

Erosion and Dilation:
- for Sets
- for Functions

Gradient and Laplacian

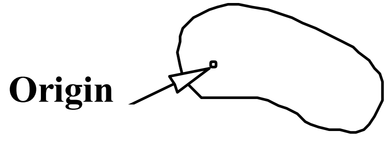
Translation invariance and Structuring Element

- A large number (not all) of morphological transforms study the lattice structure in a **translation invariant** way.

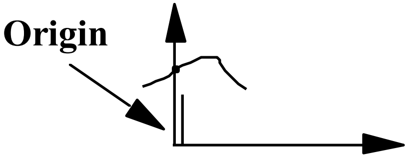
- In case of sets, since dilation commutes with union, the dilate of a set X is nothing but the union of the dilates of each of its points. Now, by translation invariance all these elementary dilates are the same, up to a translation. The operation is thus characterized by the transform of the origin, called **structuring element**.

- By the same way, in case of functions, the property of (spatial and vertical) translation invariance generates the notion of **a structuring function**.

Examples of structuring elements:



Structuring set



Structuring function

Set Translation Invariance

- Suppose set E equipped with a translation **τ** .The operations ψ: $\mathcal{P}(E) \rightarrow \mathcal{P}(E)$ which are translation invariant are called **τ-applications** .

- Then, the two basic dilations on $\mathcal{P}(E)$ are:
  - The **Minkowski Addition** , which is the **unique** τ-dilatation,
  - The **Geodesic Dilation**, which is limited to a given mask .

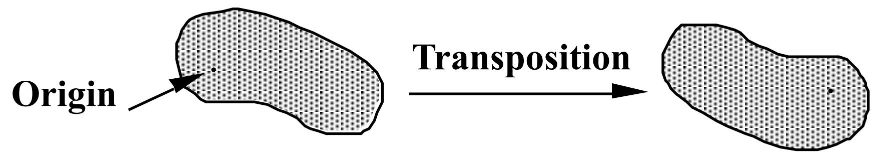
- for all $X \subseteq E$, introduce:

  1) set $\check{X}_B$, **translate** of X according to vector b :

  $$\check{X}_B = \{x+b,\ x \in X\}$$

  2) set X , **transposed** or **reflected** of X :

  $$X = \{ -x\ ,\ x \in X\}$$



we have: $x \in \check{B}_z \Leftrightarrow z - x \in B$. Note that B is **symmetrical** when it is equal to its transposed.

Set Dilation and Minkowski Addition

- We have seen that every Minkowski Addition is a dilation δ_B characterized by its Stucturing Element B. $\delta_B(X)$ is denoted by $X \oplus B$. We have :

$$X \oplus B = \cup \{ B_x , \ x \in X \}$$

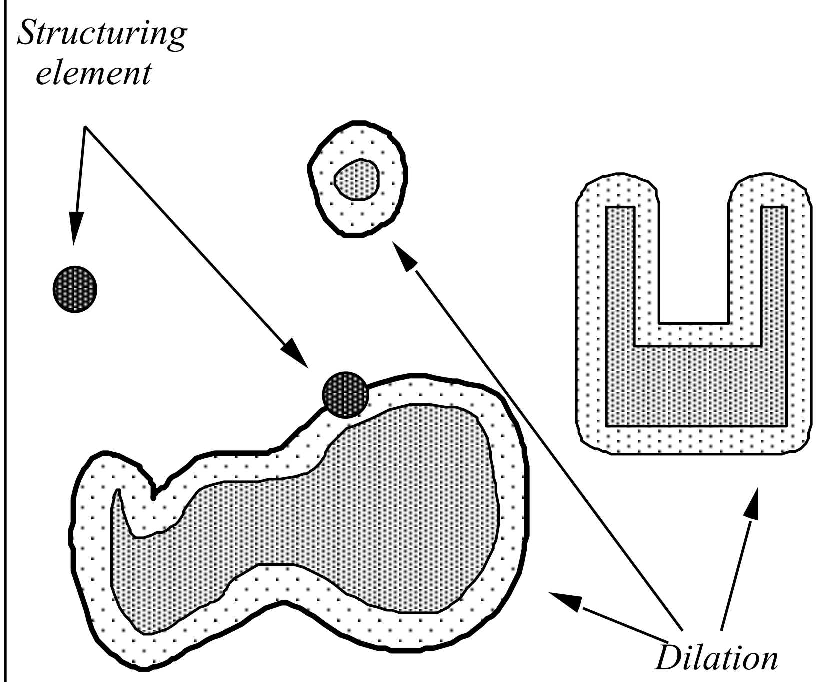
$$= \{ x + b, \ x \in X, \ b \in B \} = \cup \{ X_b, \ b \in B \}$$

Now, $z \in \delta_B(X) \Leftrightarrow \{ b = z\text{-}x \in B \ \text{ et } \ x \in X \}$

$$\Leftrightarrow \quad \{ \exists x: x \in \check{B}_z \cap X \}$$

- Hence the dilate of X by B is the locus of those points z such that the transposed set $\check{B}_z$ hits X :

- $\delta_B(X) = \{ z: \ \check{B}_z \cap X \neq \emptyset \}$



Set Erosion and Minkowski Substraction

- The Minkowski substraction of X by B is the erosion $X \ominus B$ ***adjoint*** to $X \oplus B$.

- It turns out to be the locus of the positions of the centre z of the structuring element B_z when the latter is included in X :

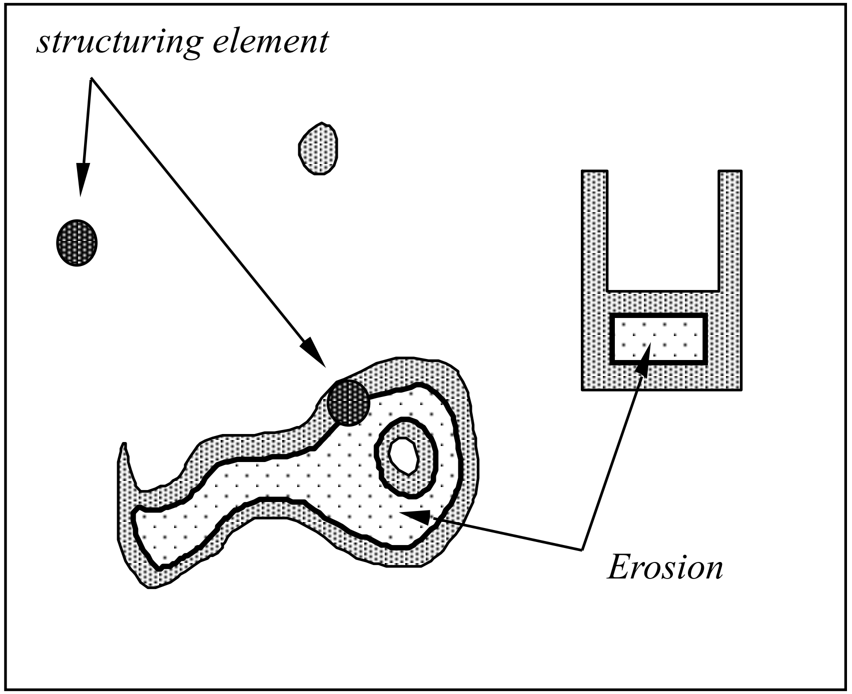
$$\varepsilon_B(X) = X \ominus B = \{ z : B_z \subset X \}$$

Now :

$$B_z \subset X \Leftrightarrow \forall b \in B: b+z \in X \Leftrightarrow \forall b \in B: z \in X_{-b}$$

hence :

$$X \ominus B = \cap \{ X_b, b \in \check{B} \}$$



Properties of Minkowski Operations (I)

1) **Increasingness and distributivity**:

$$X \subset Y \Rightarrow \begin{cases} \varepsilon_B(X) \subset \varepsilon_B(Y) \\ \delta_B(X) \subset \delta_B(Y) \end{cases} \qquad \begin{aligned} \varepsilon_B(X \cap Y) &= \varepsilon_B(X) \cap \varepsilon_B(Y) \\ \delta_B(X \cup Y) &= \delta_B(X) \cup \delta_B(Y) \end{aligned}$$

2) **Adjunction** :

$$X \subseteq \varepsilon_B(Y) = X \ominus B \quad \Leftrightarrow \quad \delta_B(X) = X \oplus B \subseteq Y$$

3) **Semi-group**: The composition product of two dilations (*resp*. erosions) is still a dilation (*resp*. erosion). Indeed, we have:

$$\delta_{B2}\delta_{B1}(X) = \cup \{ B_2(y) , y \in \cup \{ B_1(x) , x \in X \} = \cup \{ \delta_{B2}[B_1(x)] , x \in X \} ,$$

which results in the non commutative rule:

$$\delta_{B2}\delta_{B1} = \delta_A \ ; \quad \varepsilon_{B2}\varepsilon_{B1} = \varepsilon_A \quad \text{with} \quad A = \delta_{B2}(B_1)$$

4) **information**: Semi-group ⇒no inverse ⇔ dilation and erosion can only **loose** information.

Properties of Minkowski Operations (II)

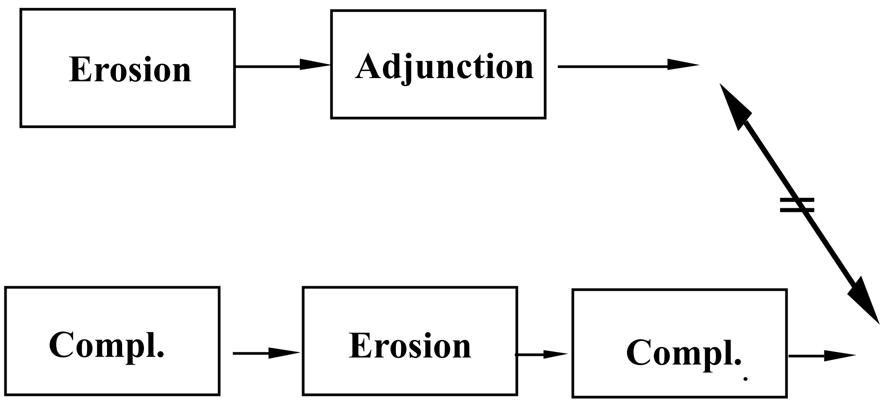
5) Duality under complement:

Let ψ^* be the dual version of ε_B for complément

• $\psi^*(X) = [\varepsilon_B(X^c)]^c = [\cap\{(X_b)^c, -b \in B\}]^c$

i.e.: $\psi^*(X) = \cup\{X_b, -b \in B\} = \delta_B(X)$

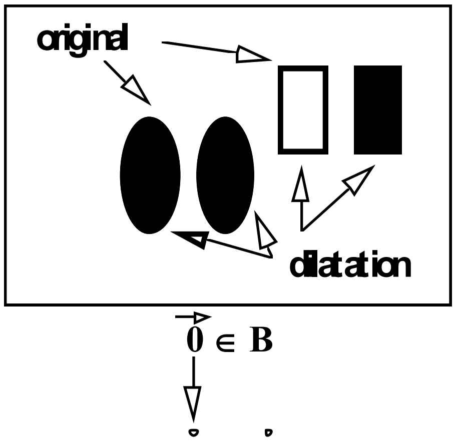
(*we find the adjonction iff B is symmetrical*)

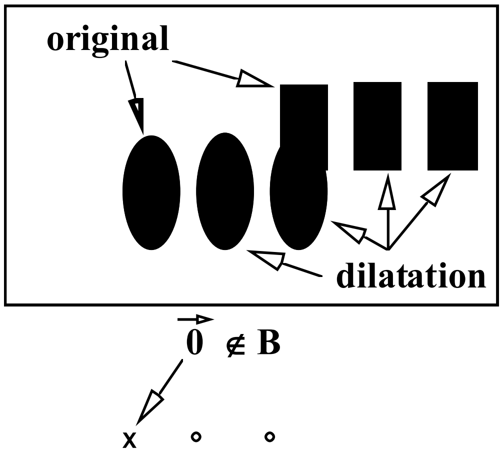


6) Extensivity :

$$\text{si } \vec{0} \in B \Rightarrow \begin{cases} \varepsilon_B(X) \subset X \\ X \subset \delta_B(X) \end{cases}$$

Dilation is extensive and erosion is anti-extensive if and only if **B contains the origin**





Properties of Minkowski Operations (III)

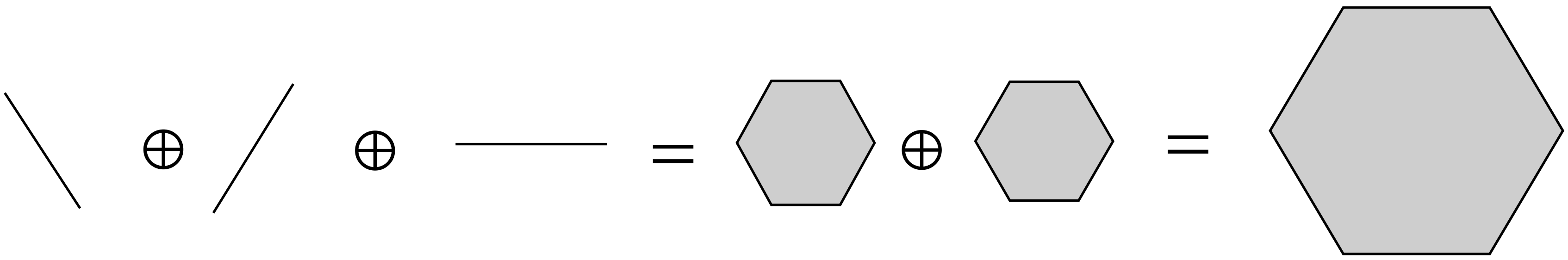
7) **Convex Structuring Elements** :

- In the Euclidean space R^n denote by λB the homothétics of B by factor λ. The semi-goup law:

$$[(X \oplus \lambda B) \oplus \mu B)] = X \oplus (\lambda + \mu) B$$

is satisfied if and only if B is **compact convex** $(x,y \in B \Rightarrow [x,y] \in B)$. Moreover, if B is plane and symmetrical, it is equal to a product of dilations by **segments**.

- Practically, the dilation (*resp*. the erosion) of a set X by the convex structuring element λB reduces to λ dilations (*resp*. erosions) by the structuring element B. Itération acts as a magnification factor.

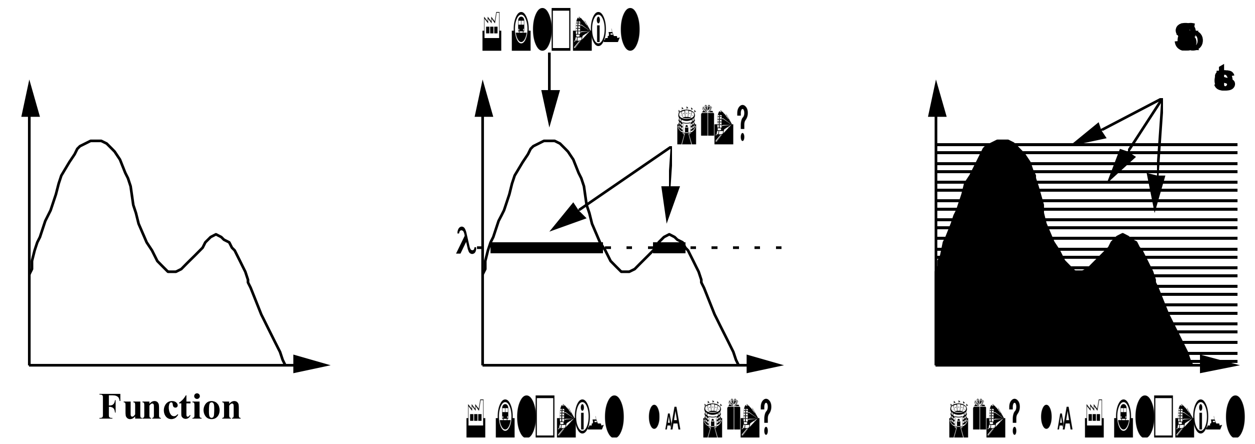


Equivalence between Sets and Functions

A function can be viewed as a **stack of decreasing sets**. Each set is the intersection between the function and a plane of constant level.

$$X_f(\lambda) = \{x \in R,\ f(x) \geq \lambda\} \quad \Leftrightarrow \quad f(x) = \mathrm{Sup}\{\lambda \text{ such that } x \in X_f(\lambda)\}$$

If f is a function, the following inclusion holds: $\forall\ \mu \leq \lambda \in R,\ \ X_f(\lambda) \subset X_f(\mu)$

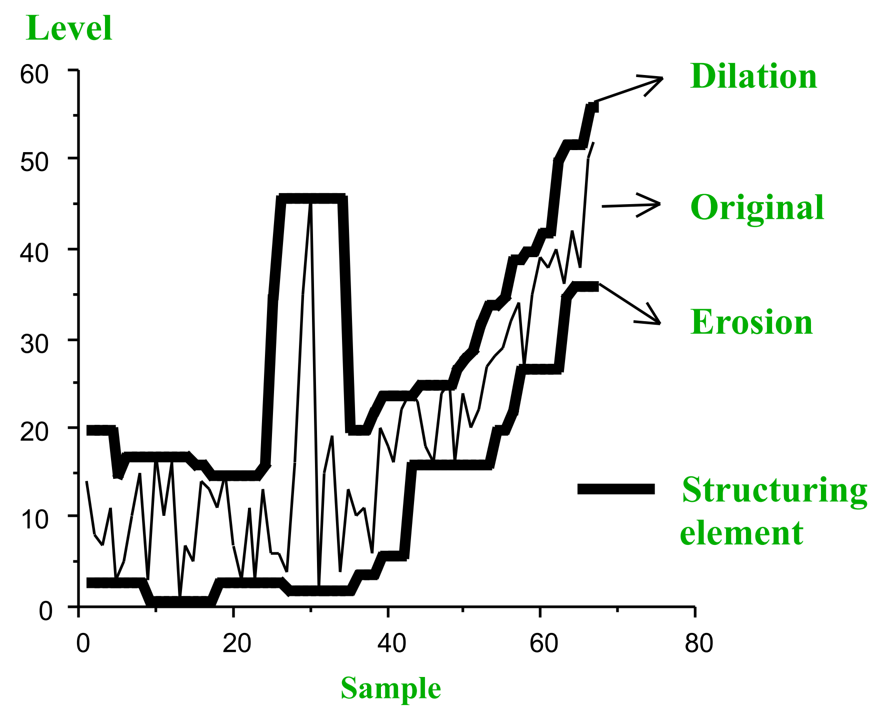


Dilation and Erosion by a flat structuring Element

Definition: The dilation (erosion) of a function by a flat structuring element can be defined as the dilation (erosion) of each set $X_f(\lambda)$ by a set B. This definition leads to the following formula:

$$X \ominus B = \varepsilon_B(f(x)) = \inf_{y \in B} [f(x-y)]$$

$$X \oplus B = \delta_B(f(x)) = \sup_{y \in B} [f(x-y)]$$



- The erosion shrinks positive pics. Pics thinner that the structuring element disappear.
- The dilation expands positive pics.
- Effects on negative pics are dual (the erosion expands them, the dilation shrinks them).

Morphological Gradients

Goal:

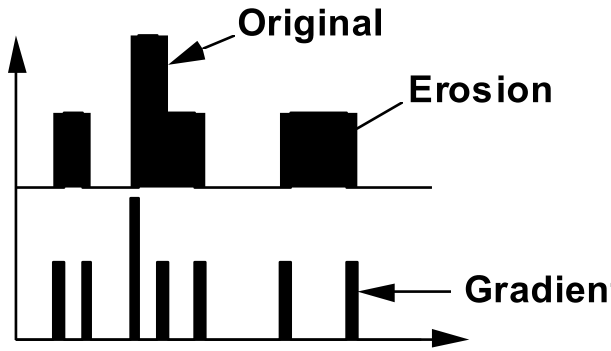
The goal of gradients transformations is to highlight contours. In digital morphology, three gradients based on **the unit disc** are defined:

Gradient by erosion:

- It is the residue between the **identity** and an **erosion ,** *i.e.*:

for sets $g^{-}(X) = X \setminus \varepsilon(X)$

for functions $g^{-}(f) = f - \varepsilon(f)$

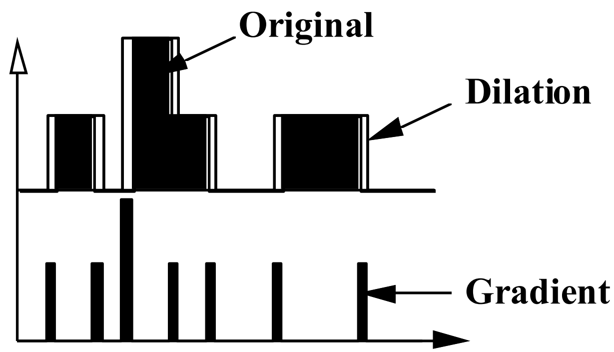


Gradient by dilation:

- It is the residue between a **dilation** and the **identity**, *i.e.* :

for sets $g^{+}(X) = \delta(X) \setminus X$

for functions $g^{+}(f) = \delta(f) - f$



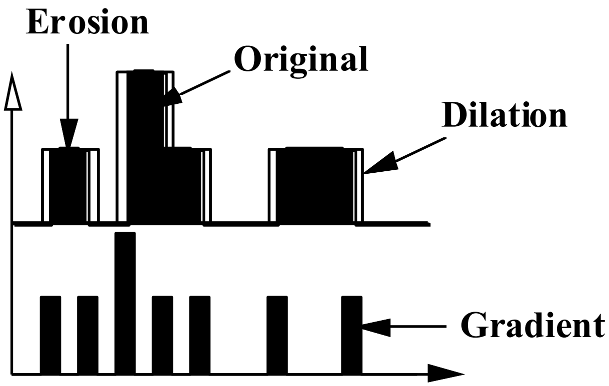
Morphological Gradients (II) and Laplacian

Symmetrical gradient:

- It is the residue between a **dilation** and an **erosion**:

for sets $g(X) = \delta(X) \setminus \varepsilon(X)$

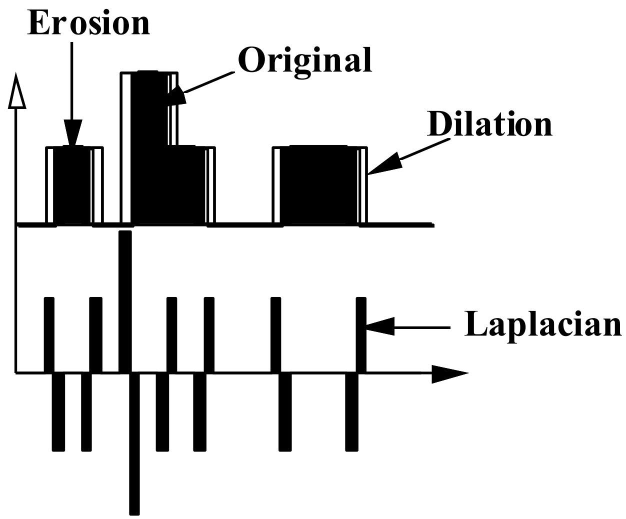
for functions $g(f) = (\delta(f) - \varepsilon(f))/2$



Laplacian:

- It is the **residue** between the **gradients** by dilation and erosion, for functions :

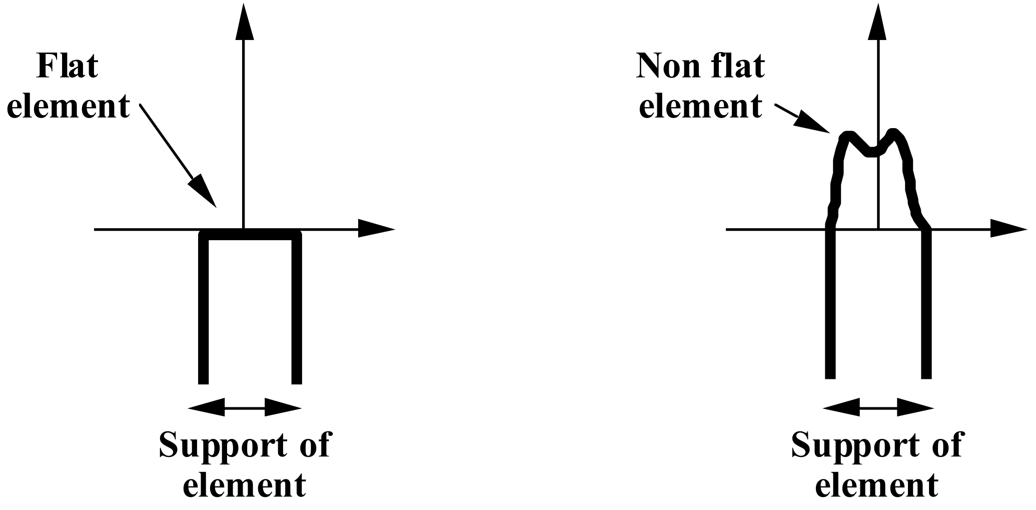
$$L(f) = g^+(f) - g^-(f)$$



Note: These notions correspond the "classical" notions of gradients and laplacians (if they exist), in the limit, when the radius of disc tends towards zero.

Non Flat Structuring Elements

- Flat structuring elements can be viewed as a function of constant level, equals to 0, and whose support is the structuring set. These structuring elements can be generalized by introducing the notion of weights. The resulting elements are non flat.



Dilation and Erosion of Functions with non Flat Elements

Definition:

With non flat structuring elements, dilation and erosion are defined as:

$$X \ominus B = \varepsilon_{h(x)}(f(x)) = \inf_{y \in H} [f(x-y) - h(y)]$$

$$X \oplus B = \delta_{h(x)}(f(x)) = \sup_{y \in H} [f(x-y) + h(y)]$$

Note:

The values of the structuring element weights should have the same dimension as the signal.

Comparison with the convolution:

A parallelism between the erosion / dilation formulas and the convolution can be done

Convolution	/	Morphology
Sum.	<=>	Sup. or Inf.
Product	<=>	Sum.

$$g(x) = \delta_{h(x)}(f(x)) = \sup_{y \in H} [f(x-y) + h(y)]$$

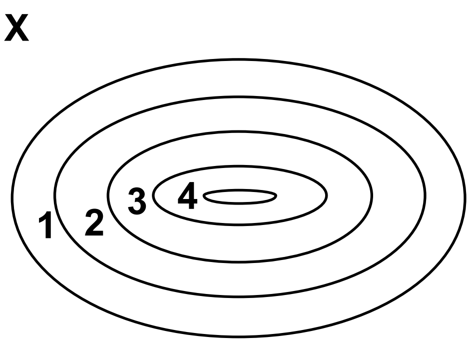
$$g(x) = h(x) * f(x) = \sum_{y \in H} f(y-x) \,.\, h(y)$$

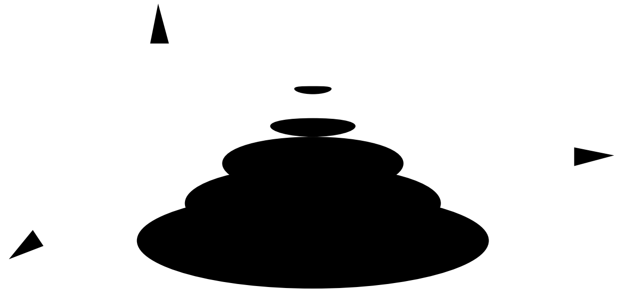
$$g(x) = \varepsilon_{h(x)}(f(x)) = \inf_{y \in H} [f(x-y) - h(y)]$$

Distance function (I)

Definition:

- The distance function is an intermediate step between sets and functions.
- When a notion of distance has been defined, It is possible to associate with each set X, its subset Xλ composed of all the points which are at a distance larger than λ from its boundary.





- When λ increases, the subsets are included within each other (and parallel in the euclidean case). They can be considered as the horizontal thresholding of a function whose grey level is λ at x if x is at a distance λ from the boundary. This function is called **Distance Function**.

Distance Function (II)

Properties:

- If the distance is characterized by the sets of disks $\delta\lambda$ of size λ, the subsets $X\lambda$ can be considered as the result of the erosions of X by the disks. More precisely:

  1) $\lambda \geq \mu \quad \Rightarrow \quad \delta_\lambda \geq \delta_\mu$

  2) $\delta_\lambda \delta_\mu \leq \delta_{\lambda+\mu}, \qquad \lambda, \mu \geq 0$

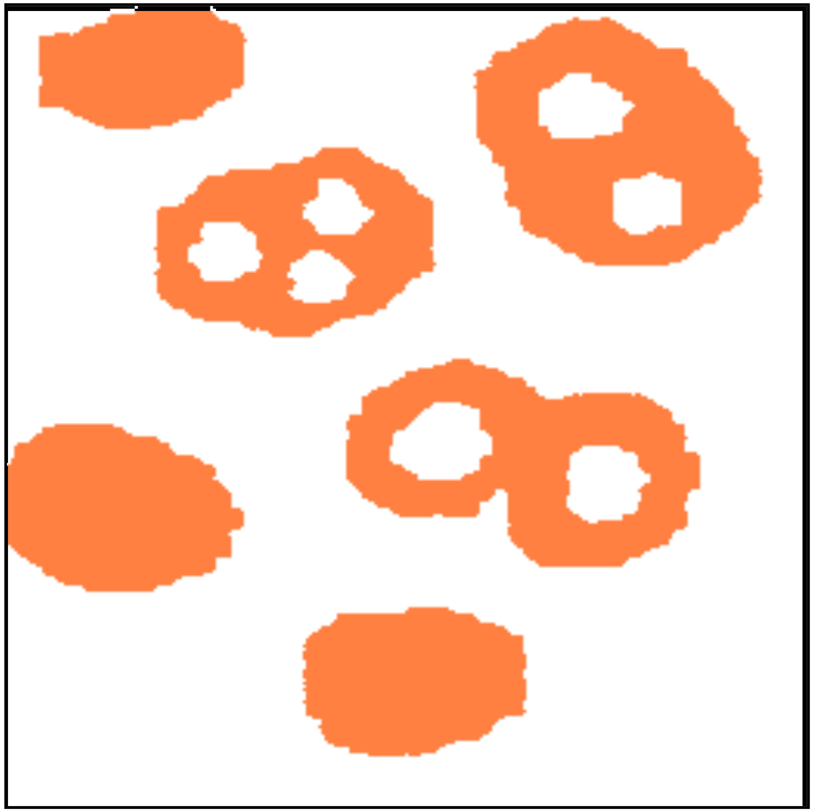
  3) $I = \wedge \{\delta_\lambda, \; \lambda > 0\}, \qquad$ I:Identity

- Conversely, each family of symmetrical dilations which fulfills these three rules defines a distance d which is characterized by:

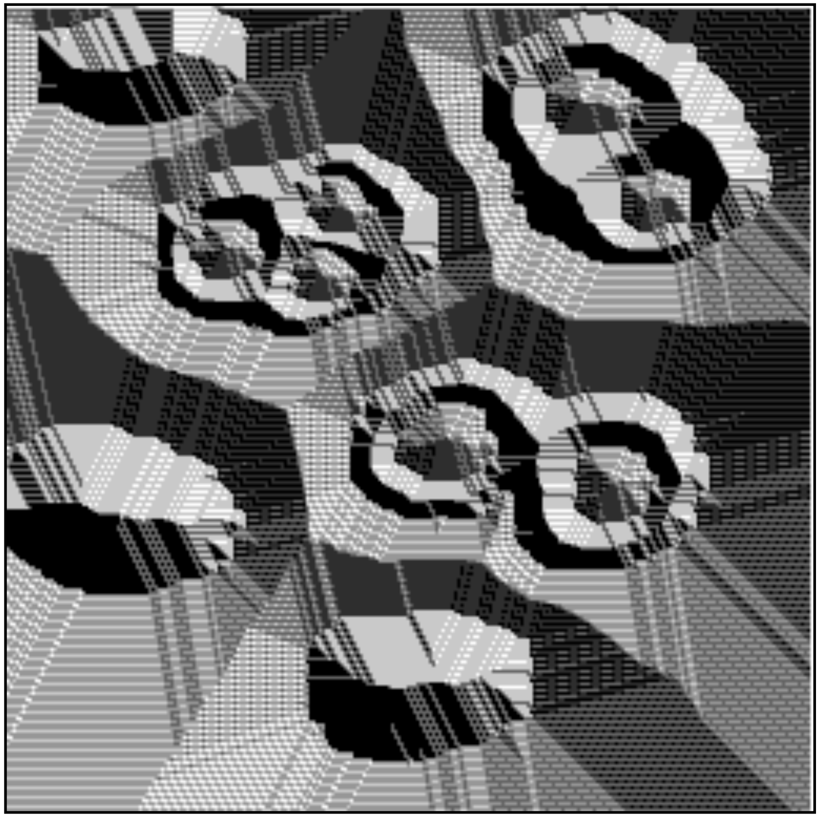
  $d(x,y) = \mathrm{Inf}\,\{x \in \delta_\lambda(y)\} = \mathrm{Inf}\,\{y \in \delta_\lambda(x)\}$

  $\delta_\lambda(y)$ is the disk of center y and radius λ.

Distance function (III): an Example



Set X

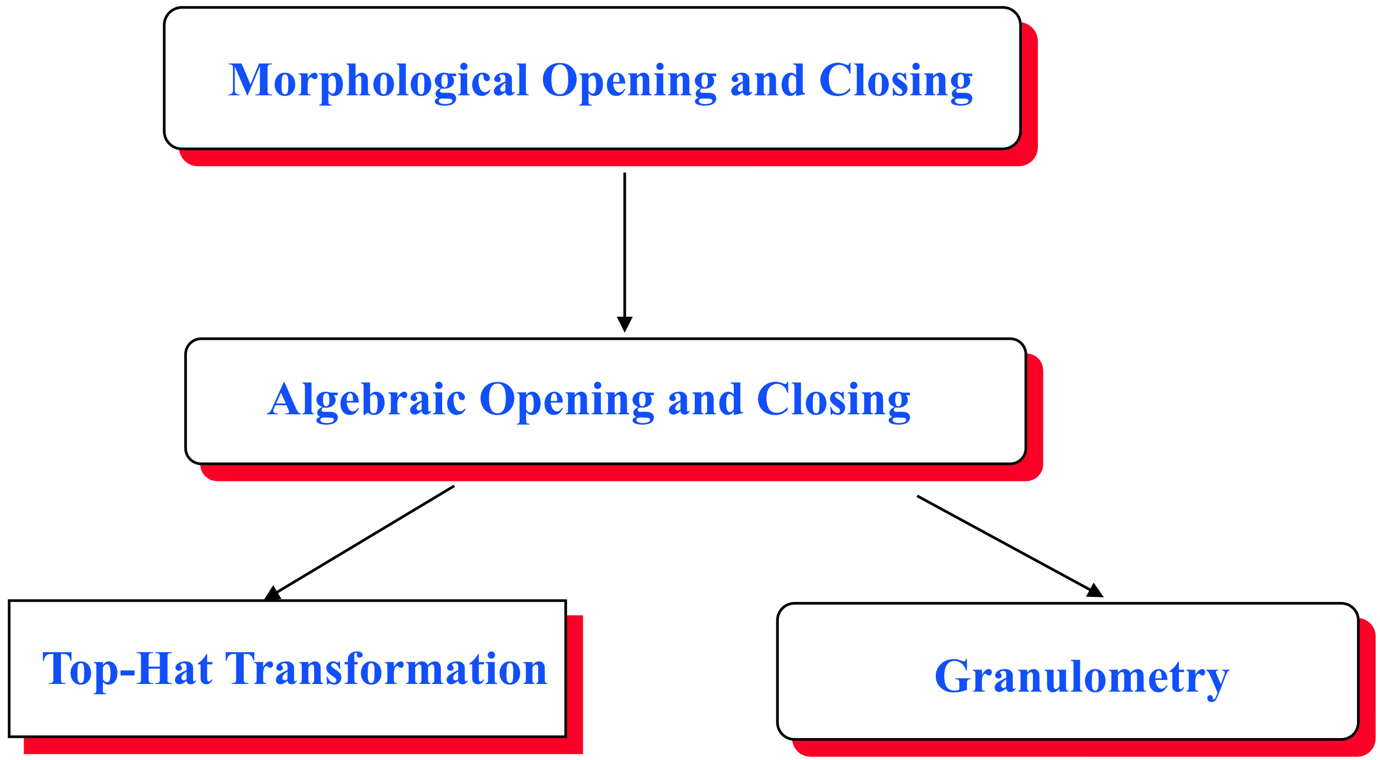


Corresponding Distance Function

Distance function (IV): another Example

The journey of Men and Women Tingary (Papunya, Australia)

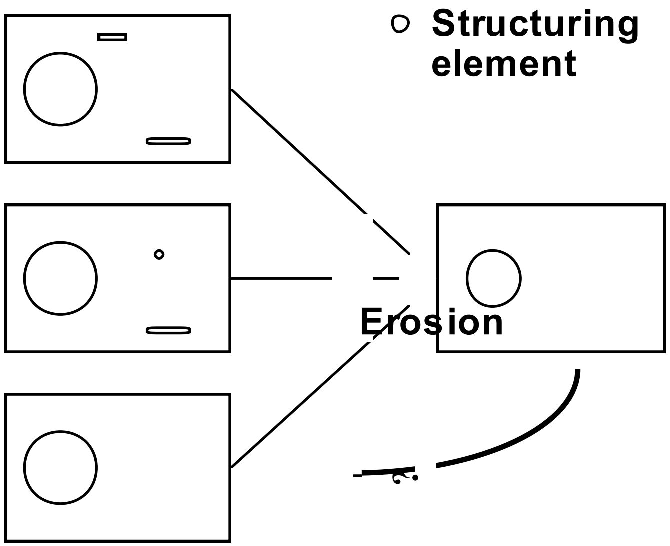
Section III: Opening, Closing



Morphological Opening and Closing

The problem of an inverse operation:

- In the classical case of linear filter, the inverse filter is simply characterized by its transfer function which is the inverse of the original filter transfer function.
- As an example, in the case of erosion, the type of nonlinearity which is involved does not allow the existence of an inverse operation. Indeed, there exist a large number of original signal which produce the same output by erosion:



However, among all possible "inverse", there is a smaller one. It is obtained by composing the erosion with the adjoint dilation.It is called morphological opening, and denoted by:

$\gamma_B = \delta_B \varepsilon_B$ *(général case)*,

$XoB = [(X \ominus B) \oplus B]$ (τ-*operators)* .

By commuting the factors δ_B et ε_B we obtain the **morphological closing :**

$\varphi_B = \varepsilon_B \delta_B$ *(général case)*,

$X \bullet B = [X \oplus B) ,B]$ (τ-*operators)*.

Properties of morphological opening and closing

Increasingness:

Opening and closing are increasing as products of increasing operations.

(Anti-)extensivity:

By doing $Y = \delta_B(X)$, and then $X = \varepsilon_B(Y)$ in adjunction $\delta_B(X) \subseteq Y \Leftrightarrow X \subseteq \varepsilon_B(Y)$, we see that:

$\delta_B\,\varepsilon_B\,(X) \subseteq X \subseteq \varepsilon_B\,\delta_B\,(X)$ hence $\varepsilon_B\,(\delta_B\,\varepsilon_B) \subseteq \varepsilon_B \subseteq (\varepsilon_B\,\delta_B)\varepsilon_B \Rightarrow$ $\varepsilon_B\,\delta_B\,\varepsilon_B = \varepsilon_B$

Idempotence:

The erosion of the opening equals the erosion of the set itself. This results in the idempotence of γ_B and of φ_B :

$\varepsilon_B\,(\delta_B\,\varepsilon_B) = \varepsilon_B \Rightarrow \delta_B\,\varepsilon_B\,(\delta_B\,\varepsilon_B) = \delta_B\,\varepsilon_B$ *i.e.* $\gamma_B\,\gamma_B = \gamma_B$ **and, by duality** $\varphi_B\,\varphi_B = \varphi_B$

Finally, if $\varepsilon_B(Y) = \varepsilon_B(X)$, then $\gamma_B(X) = \delta_B\,\varepsilon_B(X) = \delta_B\,\varepsilon_B(Y) \subseteq Y$. Hence, γ_B is the smallest inverse of erosion ε_B .

Amending Effects of the Opening

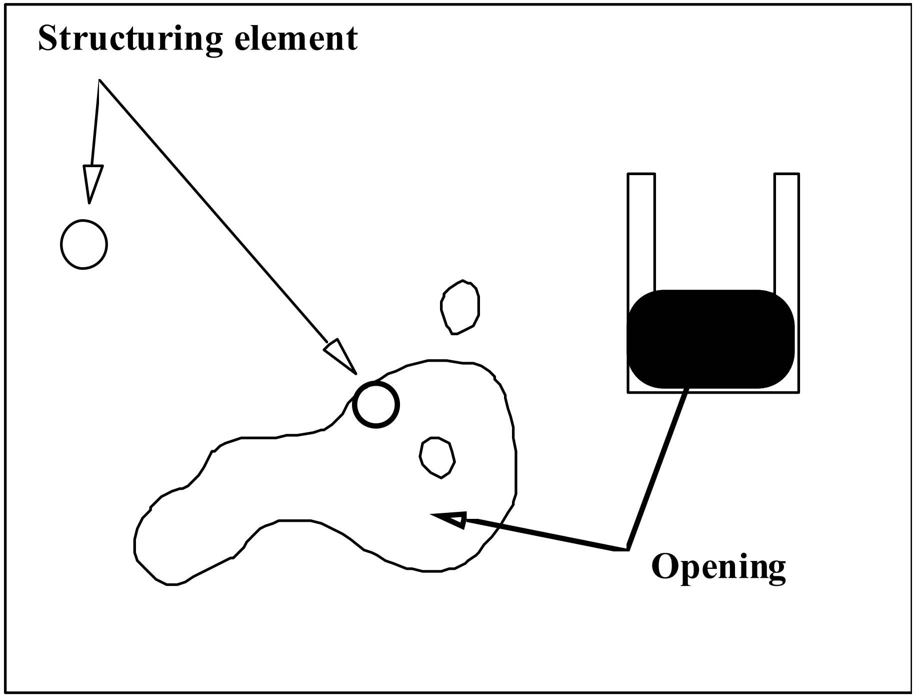
Geometrical interpretations:

$z \in \gamma_B(X) \;\Leftrightarrow\; z \in B_y$ and $y \in X \ominus B$

hence $z \in \gamma(X) \;\Leftrightarrow\; z \in B_y \subseteq X$

- the opened set $\gamma_B(X)$ is the union of the structuring elements B(x) which are included in set X.

- In case of a τ–opening, $\gamma_B(X)$ is the zone swept by the structuring element when it is constrained to be included in the set.



When B is a disc, the opening amends the caps, removes the small islands and opens isthmuses.

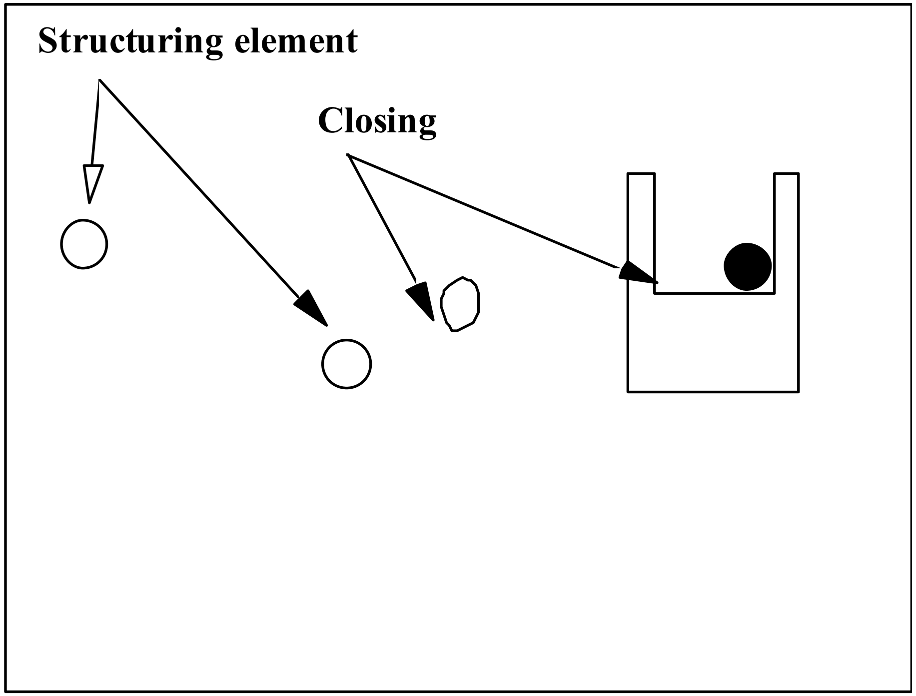
Effects of Closing on Sets

Geometrical interpretations:

- The closing is the locus of the points such that B(x) is included in the dilate $\delta_B(X)$.

- The τ-closing is the complement of the domain swept by B as it misses set X. Note that in most of the practical cases, B is symmetrical, *i.e.* identical to $\check{B}$.

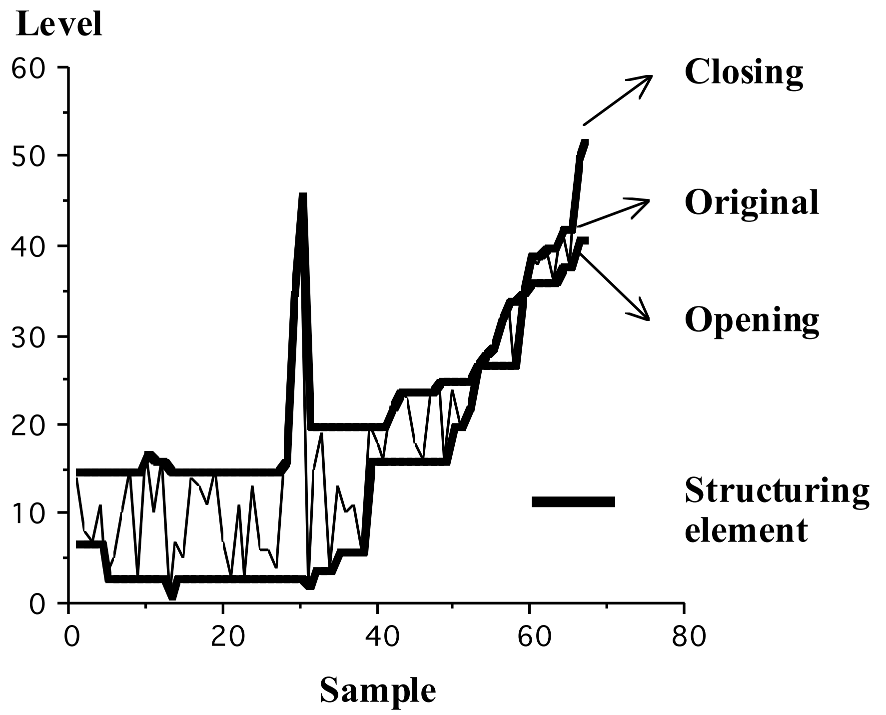
- When a shift affects erosion and dilation (because of the position of the origin), it does not acts on openings and closings.



When B is a disc, the closing closes the channels, fills completely the small lakes, and partly the gulfs.

Effects of opening and closing on functions

- The opening and closing create a simpler function than the original. They smooth in a nonlinear way.
- The opening (closing) removes positive (negative) peaks that are thinner than the structuring element.
- The opening (closing) remains below (above) the original function.



Algebraic opening and closing

The three basic properties of openings δε and closings εδ are now taken as **axioms** for the algebraic notion of openings and closings.

Definition:

In algebra, any transformation which is:
- increasing, anti-extensive and idempotent is called an (algebraic) opening,
- increasing, extensive and idempotent is called a (algebraic) closing.

Particular cases :

A large number of techniques can be used to create algebraic opening and closing. In practice, two of them are very useful:

1) Compute various opening (closing) and take the sup. of the opening (inf. of the closing).
2) Use a ***reconstruction***. process.

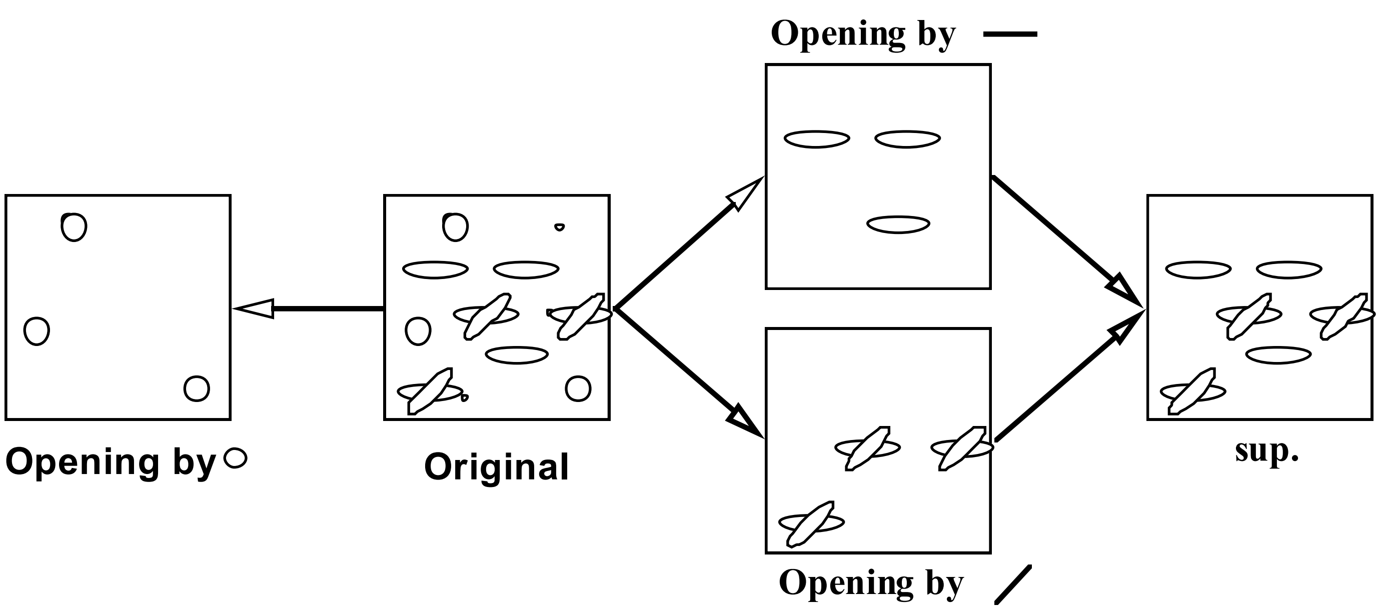
Sup. of Openings , Inf. of Closings

Theorem:

- Any sup. of opening is still an opening.
- Any inf. of closing is still a closing.

Application example:

In order to define opening with complex selection properties, one can use various morphological opening and take as final result the sup. of the opening.



"Top hat" Transformation

Goal:

- The goal of the residue by "Top hat" is to extract elements following a size or shape criterion, mainly on numerical functions.

Definition:

- The "Top hat" transformation is the difference between the **identity** and a (compatible with vertical translation) **opening** :

$$T(f) = f - \gamma(f)$$

- A dual "Top hat" can be defined: the residue between a **closing** and the **identity:**

$$T^{*}(f) = \varphi(f) - f$$

Properties of the "Top hats"

Idempotence:

- The "Top hat" is idempotent, if moreover the original signal is positive the "Top hat" is anti-extensive:

$$T(T(f)) = T(f) \qquad\qquad f > 0 \;\Rightarrow\; T(f) < f$$

Geomtrically speaking, the "Top hat" reduces to zero the **slow trends** of the signal.

Robustness:

- If Z stands for the set of points where the opening is smaller that f, *i.e.*

$$Z = \{\, x : (\gamma\, f)(x) \;<\; f(x) \,\}$$

and if g is a positive function whose support is included in Z, then we have

$$T(g) \;=\; g \qquad \text{and} \qquad T\,(f+g) = \;T(f) + T(g)$$

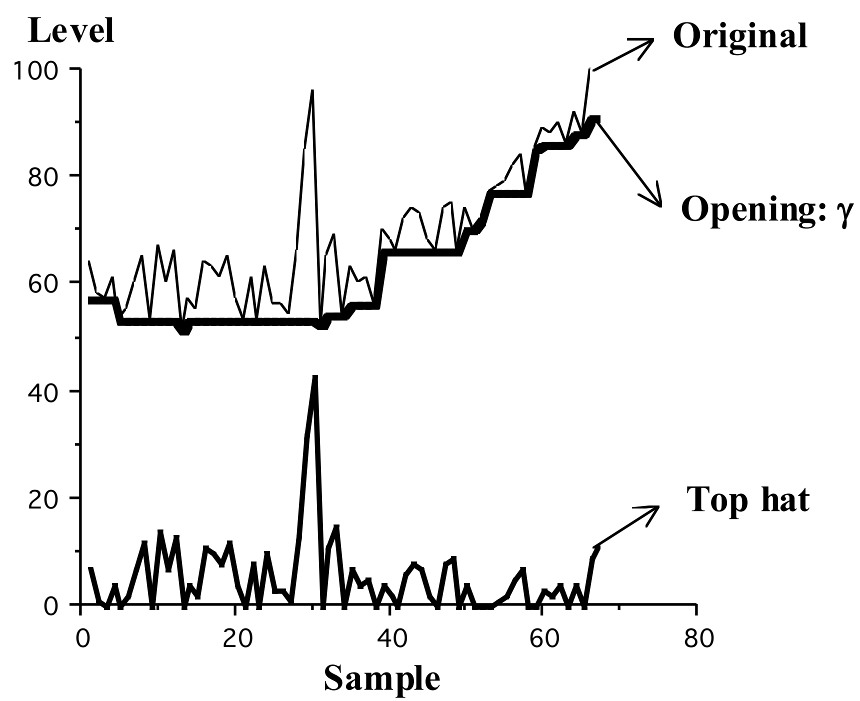
Use of the Top-hat

Sets:

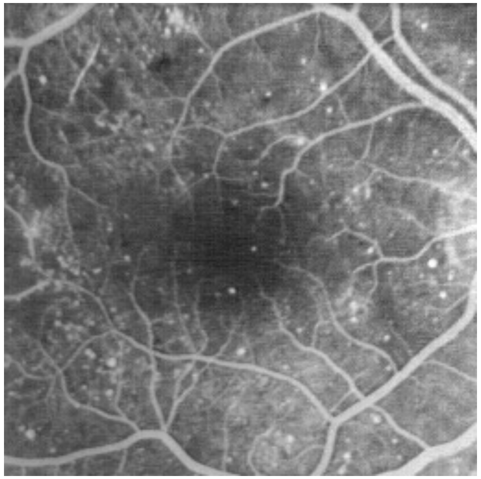
- The "Top hat" isolates the objects that have not been eliminated by the opening. That is, it removes objects larger than the structuring element.

Functions:

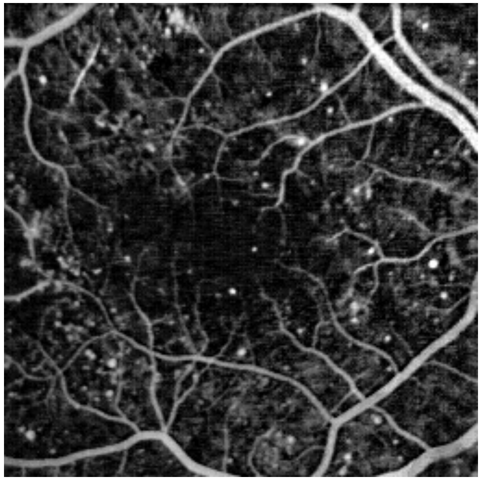
- The "Top hat" is used to extract contrasted components with respect to the background. The basic "Top hat" extracts positive components and the dual "Top hat" the negative ones.
- Intuitively, the "Top hat" compensates smooth variations of the continuous components , and thus performs a contrast enhancement.



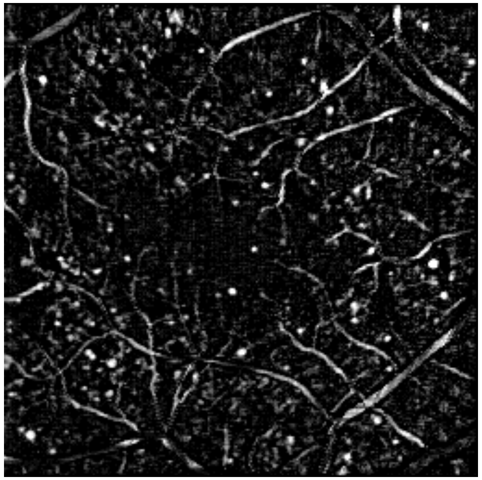
Example of Top-hat



Negative image of the retina.



Top-hat by an hexagon opening of size 10.



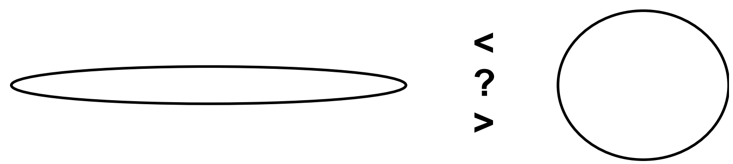
Top-hat by the sup of three segments openings of size 10.

Granulometry: an intuitive approach

- Granulometry is the study of the size characteristics of the sets and of the functions. In physics, granulometries are generally based on **sieves** ψ_λ of increasing meshes $\lambda > 0$. Now,
  - by applying sieve λ to set X, we obtain the oversieve $\boldsymbol{\psi_\lambda(X) \subseteq X}$;
  - if Y is another set containing X, the Y-oversieve, for every λ, is larger than the X-oversieve, *i.e.* $\boldsymbol{X \subseteq Y \Rightarrow \psi_\lambda(X) \subseteq \psi_\lambda(Y)}$;
  - if we compare two different meshes λ and μ such that $\lambda \geq \mu$, the μ-oversieve is larger than the λ–oversieve, *i.e.* $\boldsymbol{\lambda \geq \mu \Rightarrow \psi_\lambda(X) \subseteq \psi_\mu(X)}$;
  - finally, by applying the largest mesh λ to the μ–oversieve, we obtain again the λ–oversieve itself, *i.e.* $\boldsymbol{\psi_\lambda \psi_\mu (X) = \psi_\mu \psi_\lambda (X) = \psi_\lambda (X)}$

- Such a description of the physical sieving suggests to resort to **openings** for an adequate formalism of the size measurements.

- But what about the relations between **size** and **shape**?



Granulometry: a Formal Approach

- **Matheron Axiomatics** defines a granulometry as a family $\{\gamma_\lambda\}$
  1) of openings,
  2) depending on a positive parameter λ,
  3) and which decrease as λ increases: $\lambda \geq \mu > 0 \Rightarrow \gamma_\lambda \leq \gamma_\mu$.

- This third axiom is equivalent to the following **semi-group** law:

$$\gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda = \gamma_{\sup(\lambda,\mu)}$$

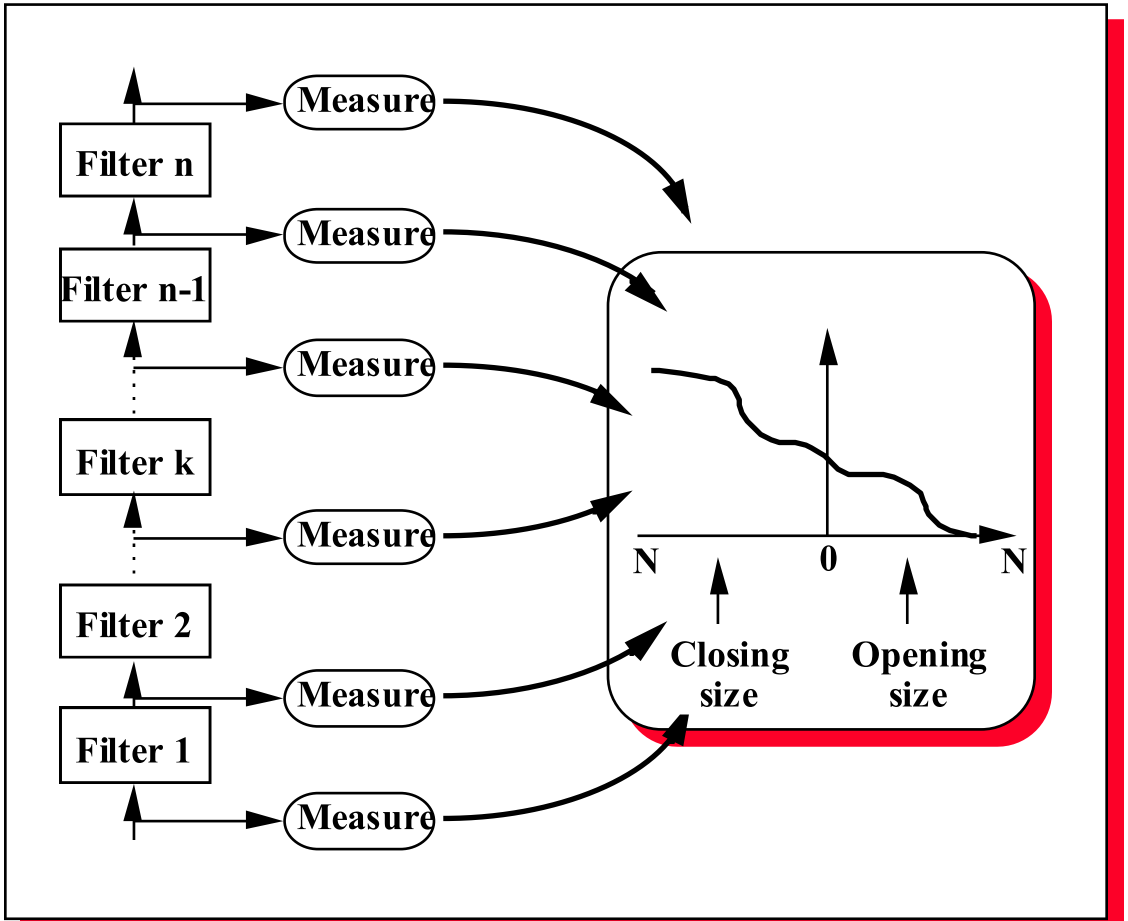
which means that the composition of two operations is equal to the stronger one.

- In practice, if we want the γ_λ's to be **morphological openings**, *i.e.* $\gamma_\lambda(X)=X\circ\lambda B$ (with homothetical structuring elements), then granulometry axioms are fulfilled if and only if the structuring element B is **convex**.

- Similarly, the families of closings $\{\varphi_\lambda , \lambda>0\}$ which are increasing in λ generate anti-granulometries. They satisfy the relationships $\varphi_\lambda \varphi_\mu = \varphi_\mu \varphi_\lambda = \varphi_{\sup(\lambda,\mu)}$.

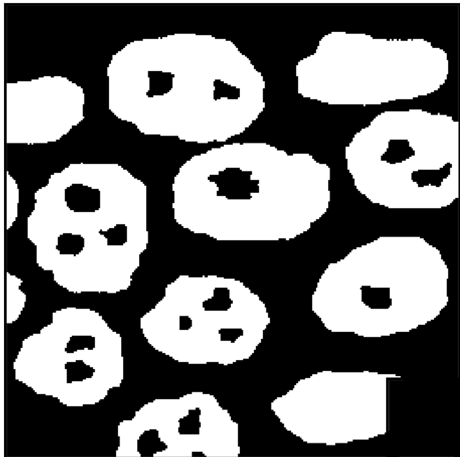
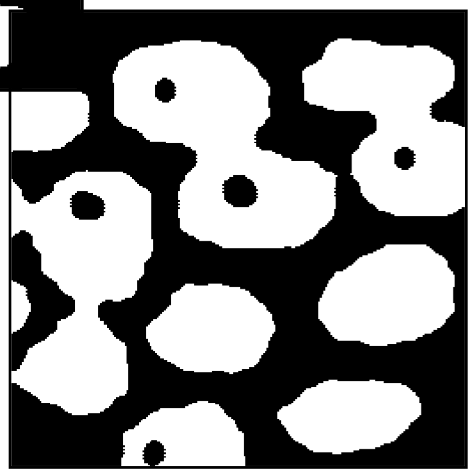
Granulometry and Measurements

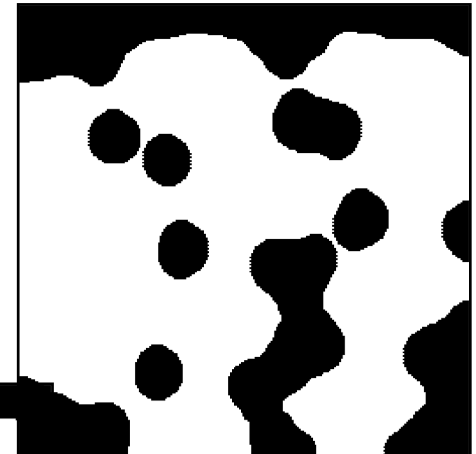
- In practice, the granulometry is computed by means of a bank of filters. They can be openings or closings.
- The filters may be constructed with structuring elements of a certain shape (segment, disc, square) of successive sizes.
- At the output of each filter, an increasing measurement is made, for example the sets area or the function integral. It results in a monotonic curve, which is normed to be a distribution function



Example of a Granulometry

From left to right, closings
by increasing discs.



Section V : Geodesy

binary geodesy :

-> metrics and dilation
-> reconstruction
-> connectivity

numerical geodesy :

-> dilation
-> reconstruction

Geodesic Transforms

Intuitive approach:

- In number of applications, the Euclidean or the digital distances from one point to another are not very useful because they do not take into account possible obstacles.
- Therefore in the framework of mathematical morphology, these distances are often replaced by the notion of a **geodesic distance**.
- Based on this new distance, it is possible to define a comprehensive class of geodesic transformations, with in particular erosion and dilation. These operations are always **isotropic,** since they bring into play balls or discs only.

Geodesic Distance

Definition:

In the set case, the geodesic distance is defined with respect to a reference set X.

dx(x,y) = Inf. of the lengths of the paths going from x to and included X;

dx(x,y) = + ∞, if no such path exits.

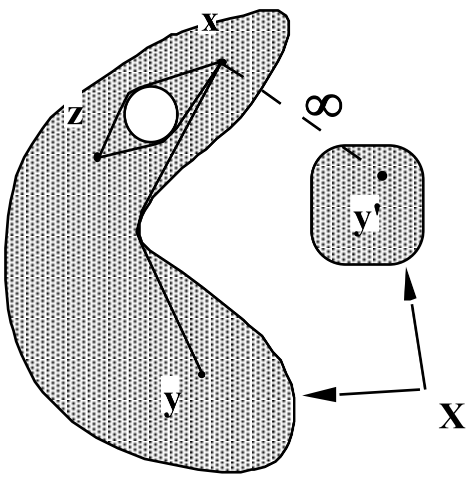
Properties:

1) It is a generalized distance:
   dx(x,y) = dx(y,x)
   dx(x,y) = 0 ⇔ x = y
   dx(x,z) ≤ dx(x,y) + dx(y,z)

2) The geodesic distance is always larger than the euclidean one;

3) A geodesic segment may not be unique.

Examples of geodesics in $\mathbf{R}^2$ *:*



N.B. the portions of geodesics included in $\overset{o}{X}$ *are line segments*

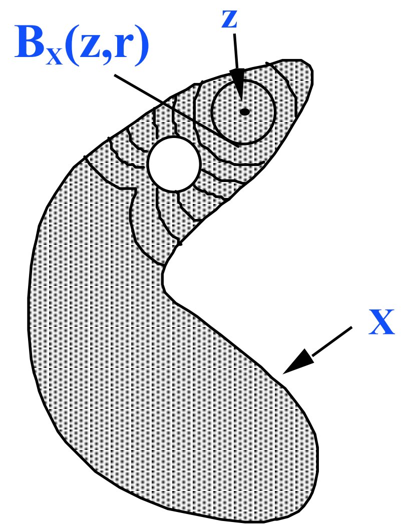
Geodesic Discs

- The notion of geodesic path is seldom used. By contrast, the notion of geodesic discs appears very often:

$$B_X(z,r) = \{y,\ d_X(z,y) \leq r\}$$

- When the radius r increases, the discs progress as a wave front emitted from z inside the medium X.

- For a given radius r, the discs B_X can be viewed as a set of structuring elements which vary from place to place.



Geodesic Dilation

- The geodesic dilation of size λ of Y inside X is written as follows:

$$\delta_{X,\lambda}(Y) = \{x \in X,\ \delta_X(x,Y) \le \lambda$$

where $d_X(x,Y) = \inf\{d_X(x,y),\ y \in Y\}$ is the geodesic distance from point x to set Y .

- As λ varies, the $\delta_{X,\lambda}$ form the additive semi groupe

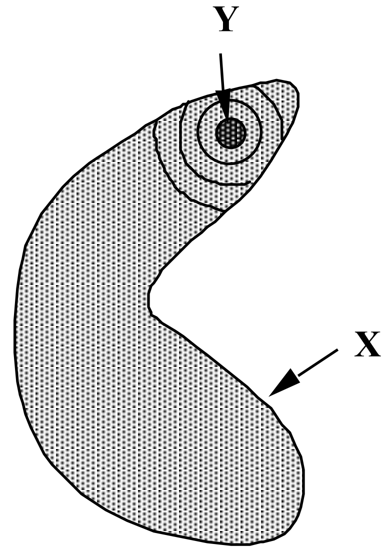
$$\delta_{X,\lambda+\mu} = \delta_{X,\lambda} [\delta_{X,\mu}] ,$$

(*useful for digital implementation).*

- Note the difference between **geodesic** and **conditional** dilations :

- $\delta_{X,\lambda}(Y) \subseteq (Y \oplus B_\lambda) \cap X$.

geodesic dilation



(Binary) Digital Geodesic Dilation

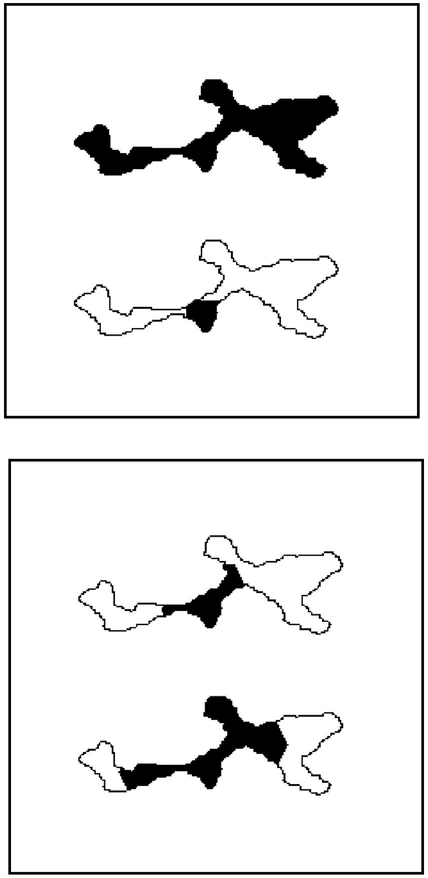
- When E is a digital metric space, and when $\delta(x)$ stands for the unit ball centered at point x, then the unit geodesic dilation is defined by the relation :

$$\delta_X(Y) = \delta(Y) \cap X$$

- The dilation of size n is then obtained by **iteration** :

$$\delta_X^{(n)}(Y) = \delta(\ldots \ \delta(\delta(Y) \cap X) \cap X \ldots) \ \cap X$$

- Note that the geodesic dilations are not translation invariant.



Geodesic Erosion for Sets

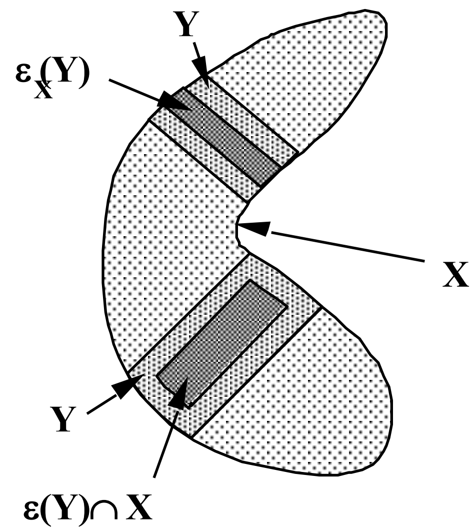
- The geodesic erosion is defined by duality with respect to the **involution within the reference** (*i.e.* $X \setminus Y = X \cap Y^C$) :
- $$\varepsilon_X(Y) = X \setminus \delta_X (X \setminus Y)$$

  i.e.:

  $$\varepsilon_X(Y) = \varepsilon (X \cup Y^c) \cap X$$

  where ε stands for Minkowski substraction
- Note that this expression is different from $\varepsilon(Y) \cap X$.

Geodesic erosion



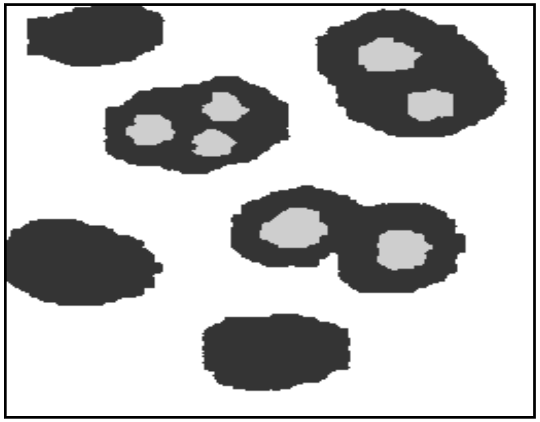
Binary Reconstruction

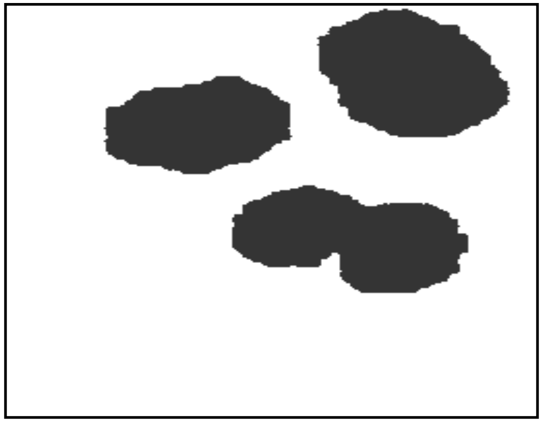
- As the grid spacing becomes finer and finer, the digital geodesic dilation tends towards the Euclidian one iff X is locally finite union of disjoint compact sets.

- In such a case, the infinite dilation of Y

$$\delta_{X,\infty}(Y) = \cup\{ \delta_{X,\lambda}(Y) , \lambda>0 \} ,$$

which is a **closing**, turns out to be the recontruction of those connected of set X that contain at leat one point of set Y .





Reconstruction Opening

- By changing our point of view, we now consider the reconstruction as an operation which holds on the (now variable) **reference X**, for a given marker Y. This operation becomes the so called **reconstruction opening** , acting on set X

opening: $\gamma^{rec}(X\,;\,Y) = \cup\{\, \delta_{X,\lambda}(Y)\,,\ \lambda>0 \,\}$

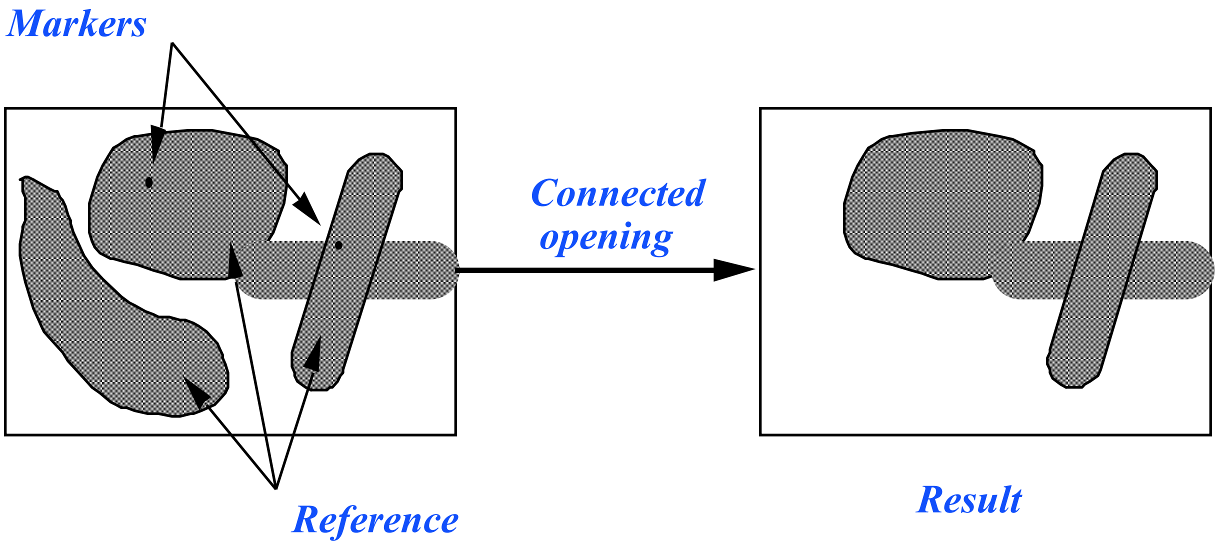
closing: $\varphi^{rec}(X\,;\,Y) = \cap\{\, \varepsilon_{X,\lambda}(Y)\,,\ \lambda>0 \,\}$

- By playing on the choices of set X and of marker Y, one can obtain various openings and residuals of interest. Here are a few examples of a common use:

Holes filling;
Ultimate erosion;
Geodesic SKIZ;
Suppression of edge touching particles;
Connected filtering;
Individual objects analysis.

Reconstruction Opening and Connectivity (I)

- All reconstruction openings are suprema of point connected ones . Therefore, they are themselve **connected** in the sense that they act only by suppressing connected components of the set under study, or of the plane sections of the function under study.



Now, the presence of a marker is not the only possible criterion: on can also keep or reject a grain according to its area, or its Ferret diameter, for example. There are no longer markers in such cases .

Reconstruction Opening and Connectivity (II)

- These situations suggest to express reconstruction openings in the following slightly more formal way:

1) Call **incrasing binary criterion** any mapping c: $\mathcal{P}(E) \rightarrow \{0,1\}$ such that:

$$A \subseteq B \quad \Rightarrow \quad c(A) \subseteq c(B)$$

2) With each criterion c associate **the trivial opening** $\gamma_T : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$

$$\gamma_T(A) = A \quad \text{if} \quad c(A) = 1$$
$$\gamma_T(A) = \varnothing \quad \text{if} \quad c(A) = 0$$

3) We vill say that γ^{rec} is **the reconstruction opening** of criterion c when :

$$\boldsymbol{\gamma^{rec} = \int\{\gamma_T \gamma_x, x \in E\}}$$

γ^{rec} acts independently on the various components of the set under study, by keeping or removing them according as they satisfy the criterion, or not.

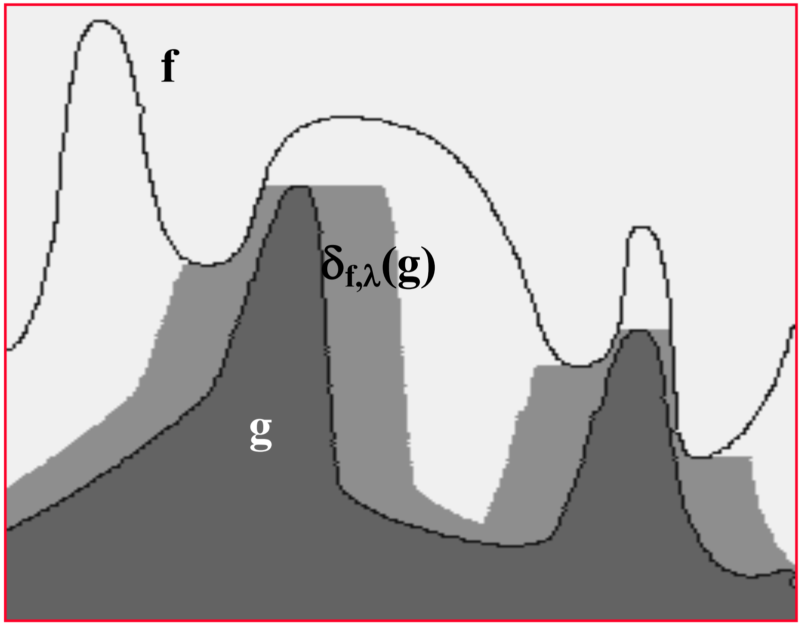
Numerical Geodesic Dilations (I)

- Let f and g be two numerical functions from R^d into R, with $g \leq f$.
- The binary geodesic dilation of size λ of each cross section of g inside that of f at the same level induces on g a dilation $\delta_{f,\lambda}(g)$.
- Equivalently, the sub-graph of $\delta_{f,\lambda}(g)$ is the set of those points of the sub-graph of f which are linked to that of g by

  - a non descending path

  - of length $\leq \lambda$.

numerical geodesic dilation of g with respect to f



Numerical geodesic Dilations (II)

- The digital version starts from the unit geodesic dilation:

$$\delta_f(g) = \inf(g \oplus B, f)$$

which is iterated n times to give that of size n

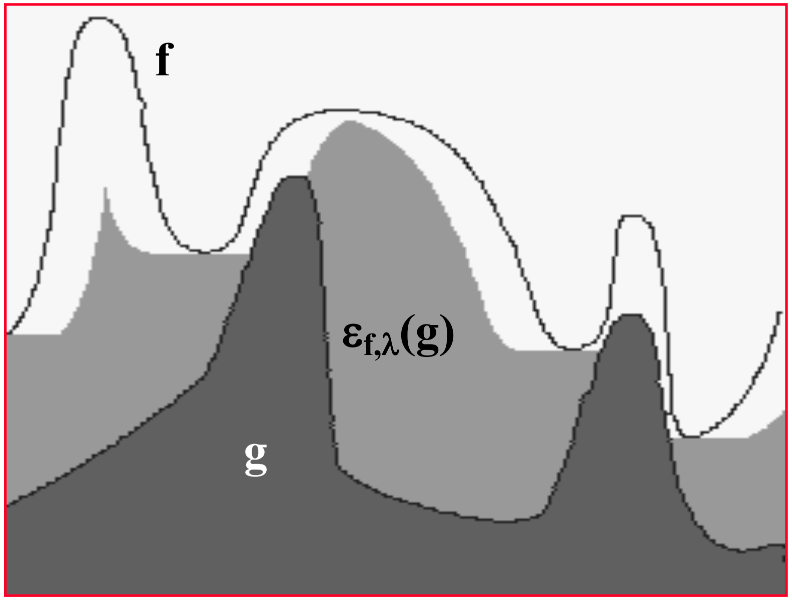
$$\delta_{f,n}(g) = \delta_f^{(n)}(g) = \delta_f(\delta_f \;....\; (\delta_f(g))).$$

- The Euclidean and digital erosions derive from the corresponding dilations by the following duality

$$\varepsilon_f(g) = m - \delta_f(m - g),$$

which is **différent** from the binary duality.

numerical geodesic erosion of f with respect to g :



Numerical Reconstruction

- The reconstruction opening of f accoding to g is the supremum of the geodesic dilations of g inside f, this sup being considered as a function of f:

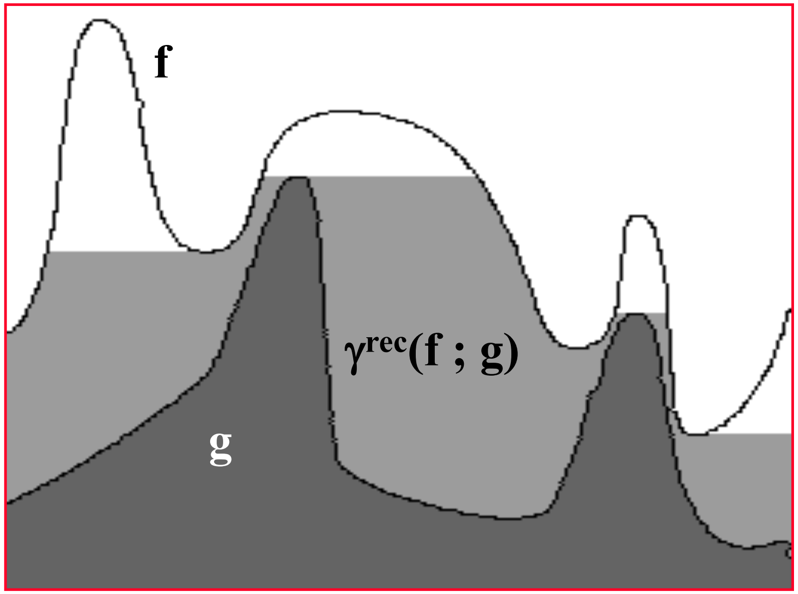
$$\gamma^{rec}(f\,;\,g) = \bigvee\{\ \delta_{f,\lambda}(g)\ ,\ \ \lambda>0\ \}$$

The dual closing for the negative is

$$\varphi^{rec}(f\,;\,g) = \ m - \gamma^{rec}(m- f\,;\,m- g)$$

- Three cases are basic for the applications :
  - swamping, *or reconstruction of a function by imposing markers for the maxima;*
  - *reconstruction from an erosion ;*
  - *contrast opening ,which extracts and filters the maxima.*

Numerical Reconstruction of g inside f :



Section VI : Applications of Geodesy

- ***Binary Geodesy :***

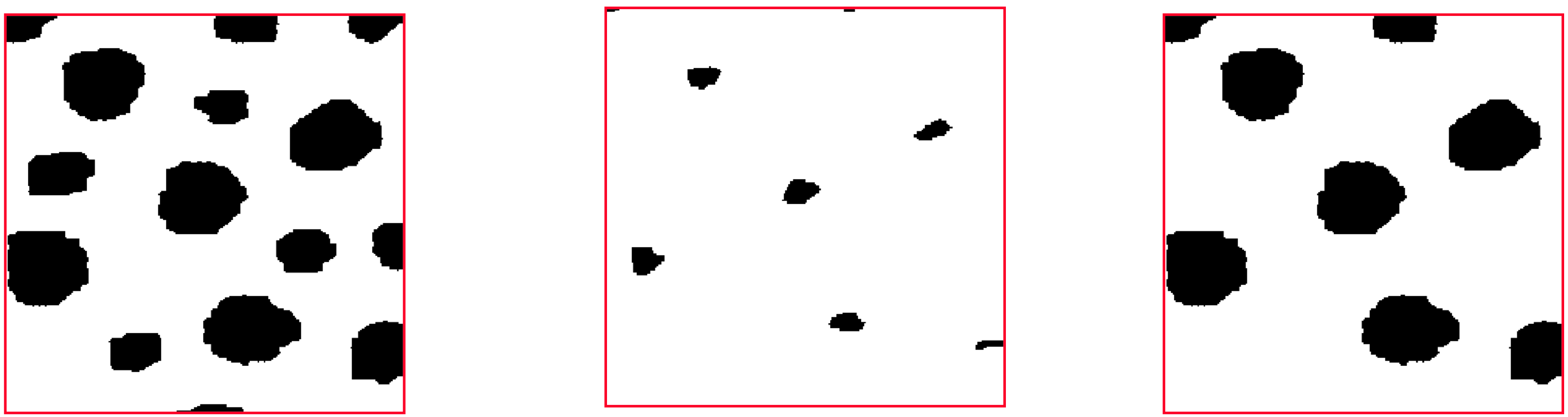
  -> edge corrections
  -> holes filling
  -> individual analysis
  -> particles extremities
  -> geodesic skiz

- ***Numerical Geodesy :***

  -> binary labelling
  -> swamping
  -> contrast opening
  -> extrema
  - > connected filters

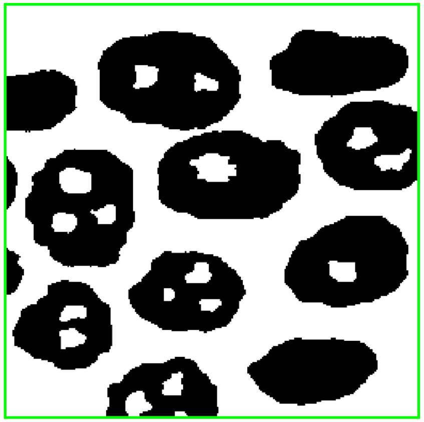
Binary Filtering by erosion-recontruction

- Firstly, the erosion $X \ominus B_\lambda$ suppresses the connected components of X that cannot contain a disc of radius λ.
- Then the opening $\gamma^{rec}(X ; Y)$ of marker $Y = X \ominus B_\lambda$ rebuilts all the others.

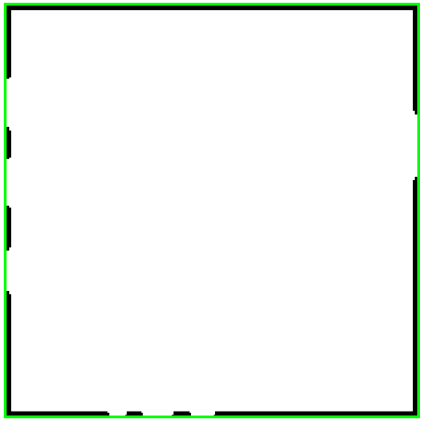


Holes Filling

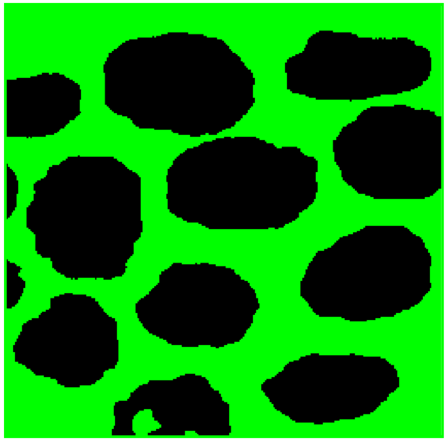
initial image
X



A = part of the edge that hits X^C



reconstruction of A inside X^C



Removal of the grains hitting the edges

- Let Z be the set of the edges , and X be the grains under study ;
- Set Z€X is reconstructed inside set X ;
- the set difference between X and the reconstruction provides the internal particles.

Individual Analysis of Particles

- **Algorithm :**

While set X is not empty do {

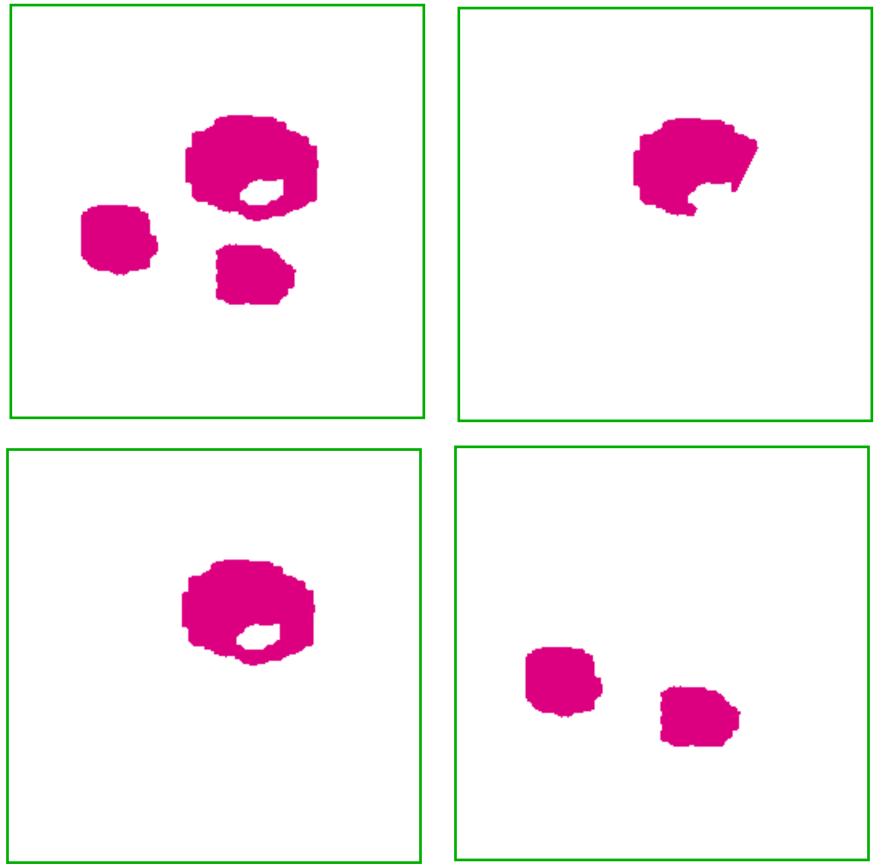
~ p := first point of the video scan;

~ Y := connected component of X reconstructed from p;

~ Processing of Y (and various measurments) ;

~ X := X \ Y

}



Extremities of a particle

- Particle X is supposed to be simply connected (*i.e.* connected and without holes) ;
- An internal centroid is provided (*for ex.* by means of thinning **Dthin**) ;
- The extremities of particle X are then defined as the geodesic ultimate erosion, inside X, of the set Y equal to X minus its centroid.

In other words, if $\varepsilon_n = \varepsilon_{X,n}(Y)$ stands for the geodesic erosion of size n of Y inside X, then

$$\textit{extremities} = \cup\ [\varepsilon_n \setminus \gamma^{rec}(\varepsilon_n ; \varepsilon_{n+1}) ,\ n \in N]$$

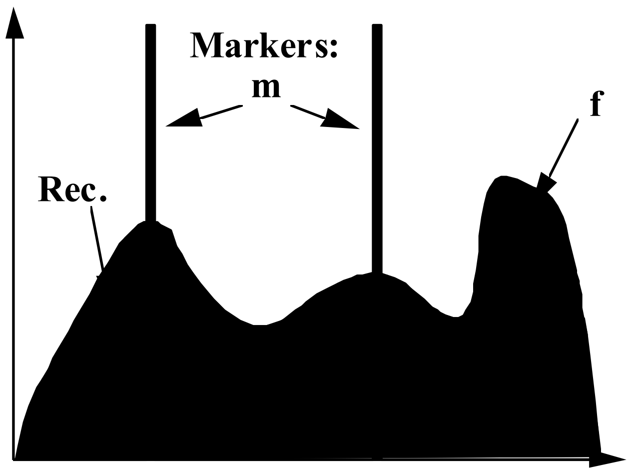
Reconstruction of a Function from Markers

Goal:

- Remove from a function the useless maxima (or minima).

Algorithm:

- The "marker" is a bi-valued (0,m) function identifying the peaks of interest.
- The reconstruction process creates a function equal to the original one in the zones of interest and eliminates maxima which are not marked.
- The result is the largest function ≤ f and admitting maxima at the marked points only. It is called the **swamping** of f (by opening).



Swamping of f by markers m
(by opening)

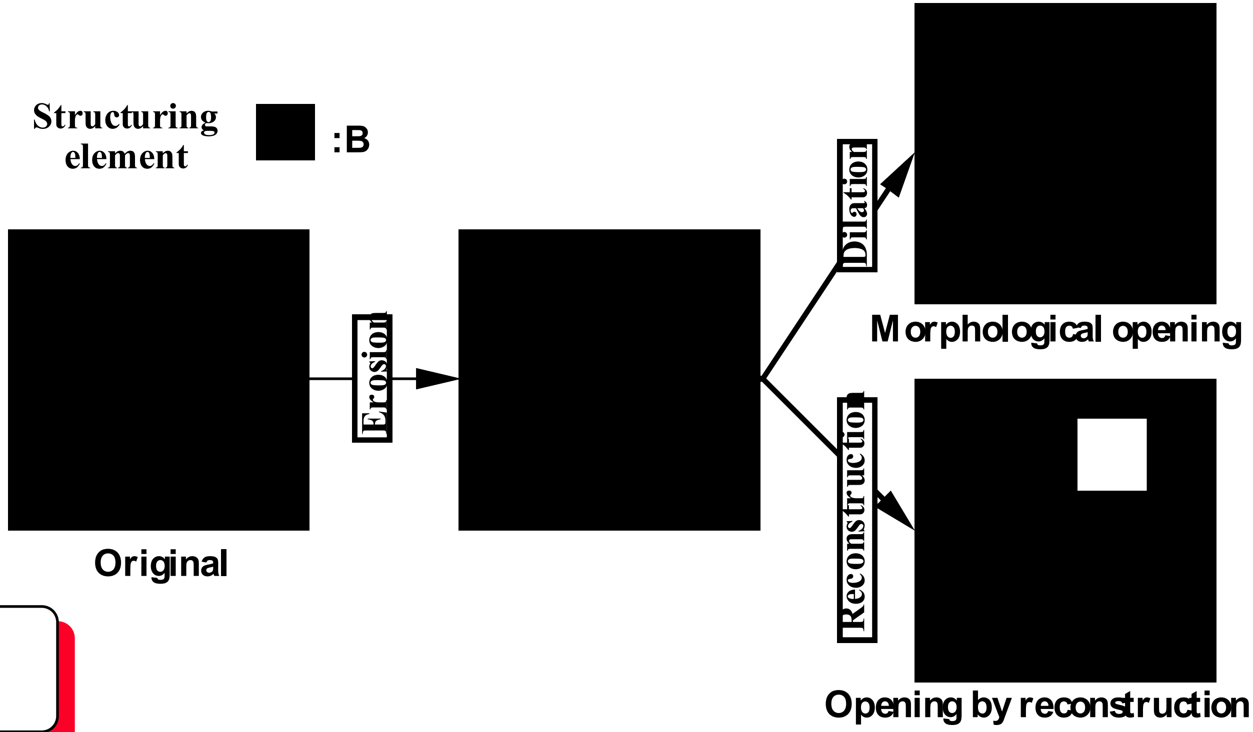
Reconstruction Opening by Erosion

Goal:

In the multidimensional case, the morphological opening modifies the various element contours. The goal of this transform is to efficiently and precisely reconstruct the contours of the objects which have not been totally removed by the filtering process.

Algorithm:

- This transformation can be expressed by means of reconstruction:
  - the mask is the original signal itself,
  - the reconstruction input is an erosion of the mask.



$$\gamma^{rec}(f,\varepsilon_B(f)) = \underset{n}{\vee}\, \delta_f^{(n)}(\varepsilon_B(f))$$

Reconstruction Opening by Dynamic

Goal :

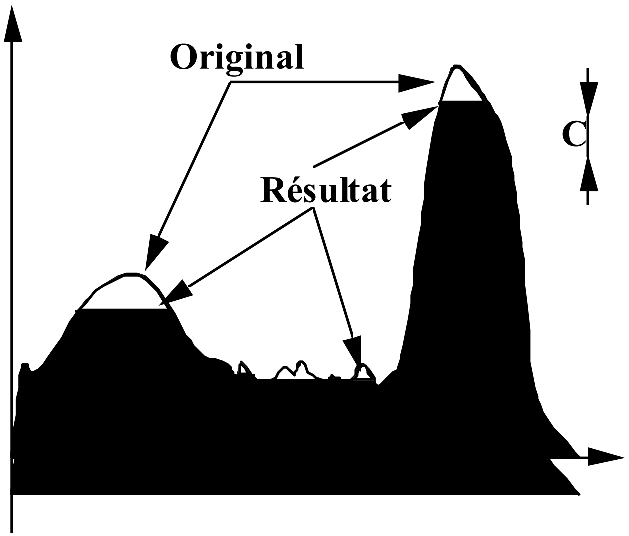
Both morphological and reconstruction openings reduce the functions according to size criteria which work on their cross sections. In opening by dynamic, the criteron holds on gray tones contrast.

Algorithm :

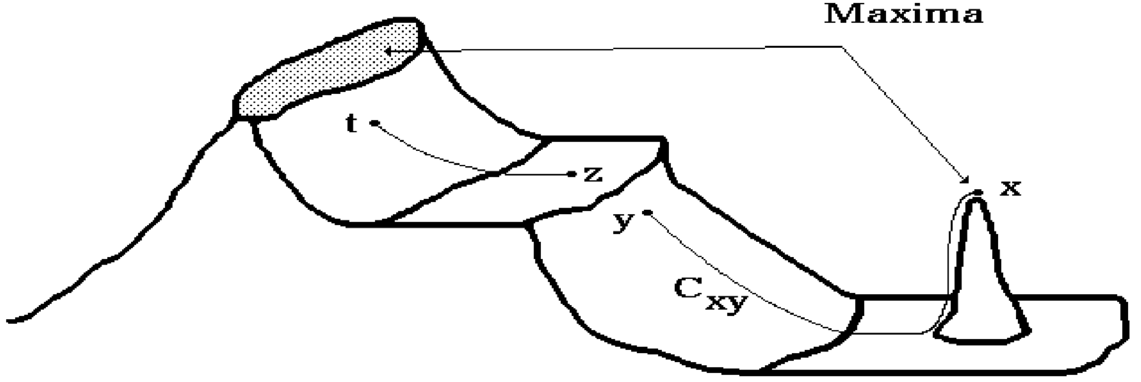
- Shift down by constant c the initial function f ;
- Rebuilt f from function f - c, *i.e.*

$$\gamma^{rec}(f,\ f\text{-}c) = \vee_n \delta_f^{(n)}(f\text{-}c)$$

- Note that the associated top-hat extracts all peaks of dynamic ≥ c



Maxima and Opening by Dynamic



- The **maxima** of a numerical function on a space E are the connected components of E where f is constant and surrounded by lower values.
- Therefore they are given by the residues of the opening by dynamic, for a shift c = 1.
- More generally, the residuals asssociated with a shift c extract the maxima surrounded by a descending zone deeper than c. They are called **Extended Maxima**

Flat Zones and Connected Operators

Definitions :

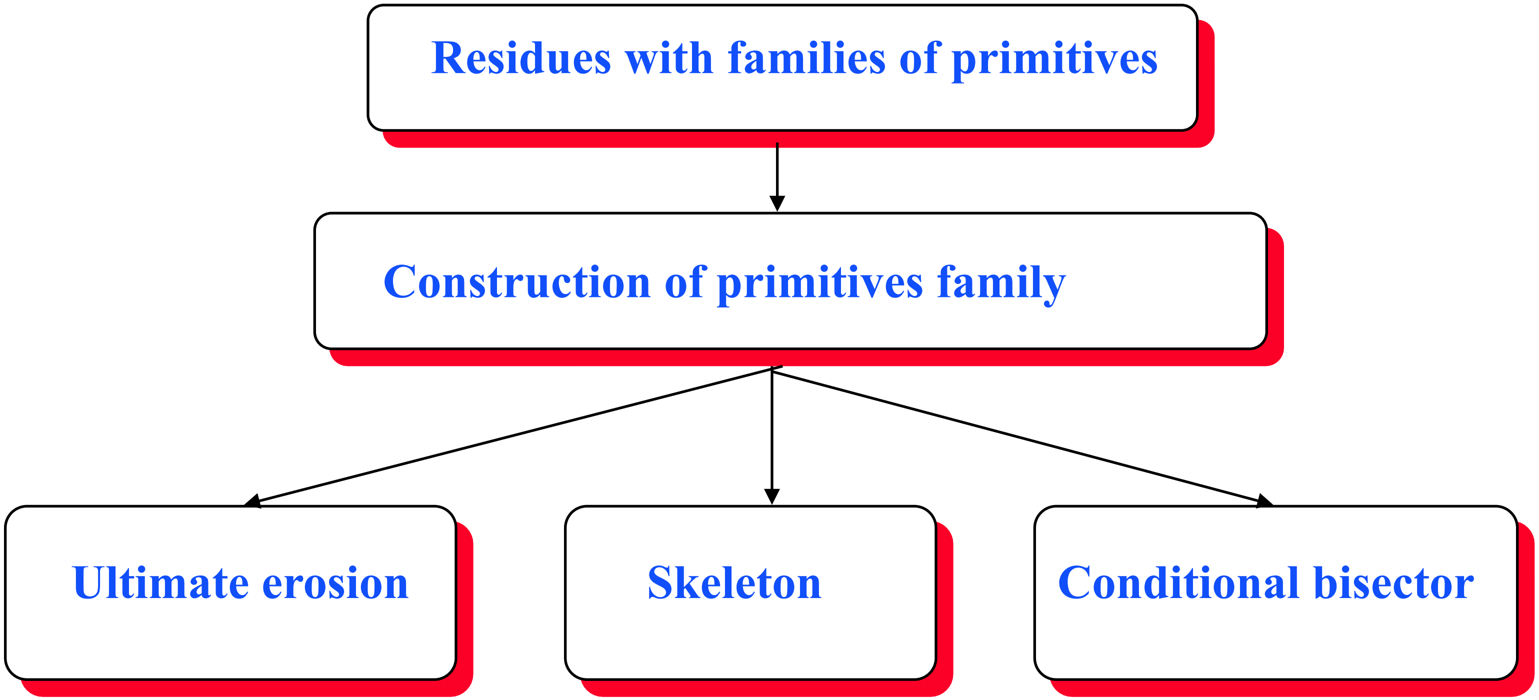
- The flat zones of a function f over a space E are the largest connected components of E where f est constant.
- An operator ψ on functions is said to be **connected** when the flat zones of ψ(f) contain thoses of f.

Properties :

Every **binary** increasing operator by reconstruction generates, via the cross sections, a connected operator on the **functions** (numerical ou multivalued ones).

- The properties of the binary operations such as to be a strong filter, to form semi-groups etc.. are systematically transmitted to the connected operators induced on the numerical functions via the cross sections .

Plan of section VII



Example of family (I): Homogeneous family

- This is the most useful case in practice. It assumes that all elementary erosions are equal. That is:

$$\forall i,j \quad \eta_i = \eta_j = \eta$$
$$\Rightarrow \quad \varepsilon_i = (\eta)^i$$

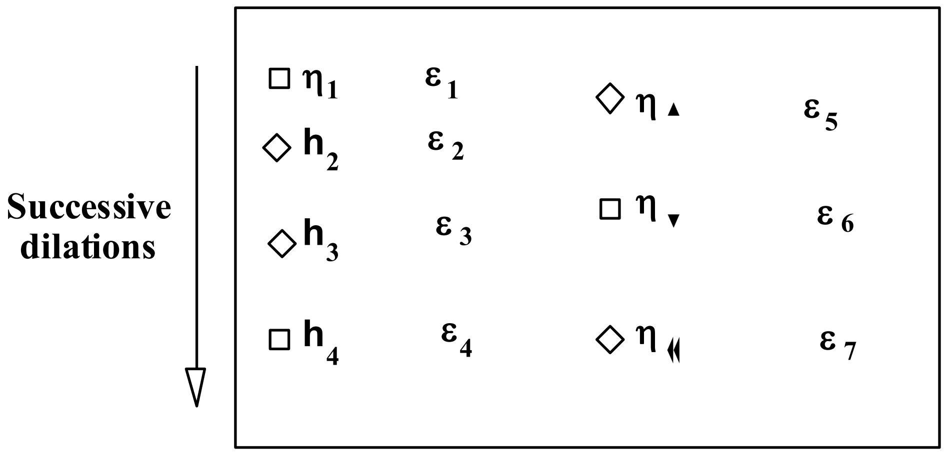
- As an example, the structuring elements are homothetics of symmetrical sets (disk, square, polygon):

Successive dilations

Example of families (II): Nonhomogeneous families

- Sometimes, it is necessary to use a finer analysis tool, that is a slower size progression of the family. In this case, the elementary erosions can vary from one order to another one.

- Example: Family of erosions with polygons such that, for each direction, the extension of each polygon side is increasing.



Ultimate erosion (I)

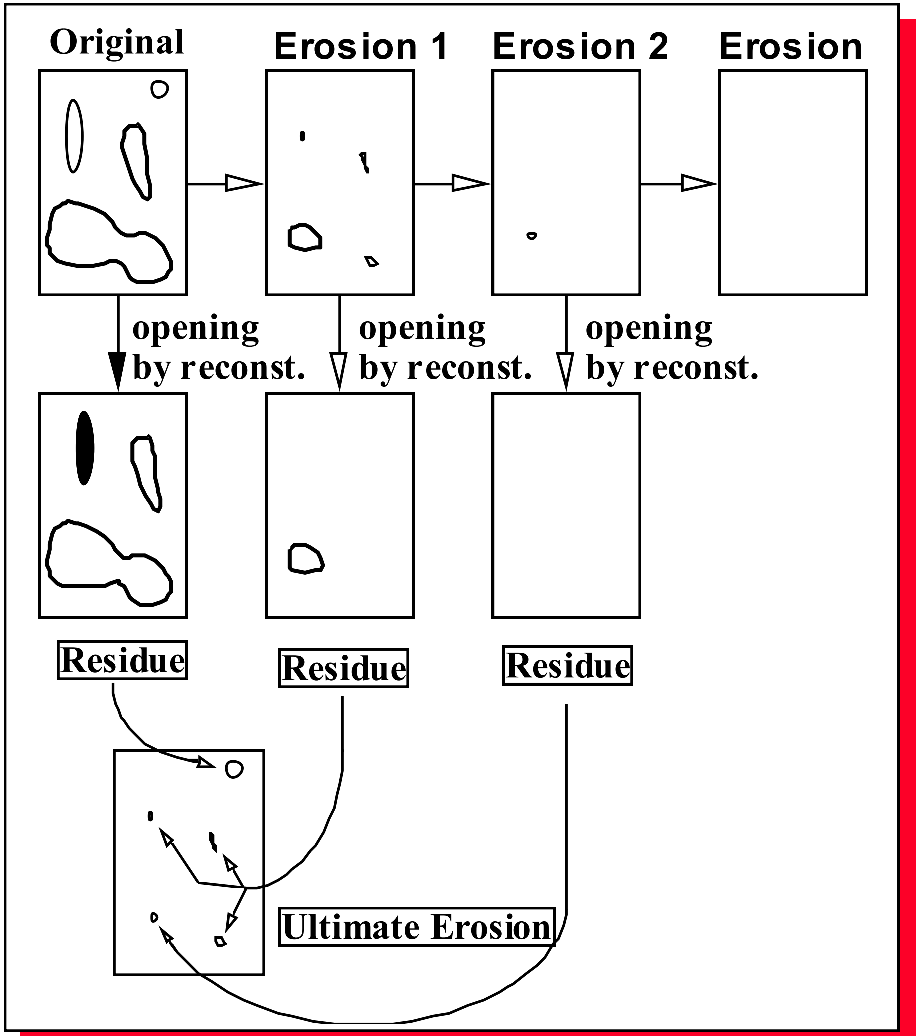
Intuitive description:

The goal is to "mark" sets. To this end, successive erosions are performed and, at each step, the total disappearance of the sets is checked. The marker is the result of the last erosion which is not empty.

In order to check if a given set is going to disappear totally in the next step, an erosion followed by a geodesic reconstruction, that is an opening by reconstruction, is performed. Two cases may occur:

1) *The erosion has removed the set:* The reconstruction result is empty.

2) *The erosion has not removed the set*: The reconstruction result is the initial set.



Ultimate erosion (II)

Definition:

- In the case of digital space, the ultimate erosion is simply the residue between the families of erosion and of opening by reconstruction of these erosions:

Residue between $\{\varepsilon_i\}$ and $\{\gamma^{rec}(\varepsilon_i, \varepsilon_{i+1})\}$

$$U(X) = \Re_{\{\varepsilon_i\},\ \{\gamma^{rec}(\varepsilon_i,\varepsilon_{i+1})\}}(X) = \bigcup_i \Re_{\varepsilon_i,\ \gamma^{rec}(\varepsilon_i,\varepsilon_{i+1})}(X) = \bigcup_i U_i(X)$$

Properties:

- The ultimate erosion is anti-extensive and the U_i are disjoint.
- $U(X)$ is thin, in the sense that its erosion by $\cup\eta_i$ is empty.
- When the family is homogeneous, the ultimate erosion is idempotent.
- If a component U_i is dilated with a structuring element i, the result is a maximal ball: It is included in the original set and and there exist no other larger structuring element in the family allowing the creation of a dilation of U_i which is included in the original set.

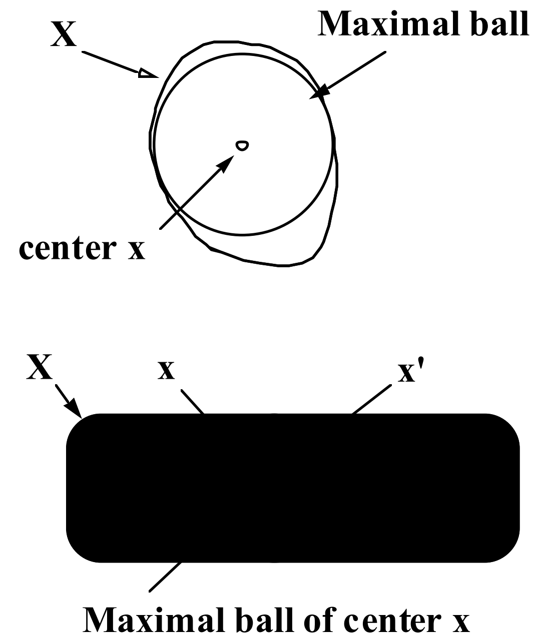
Maximal balls

- The study of ultimate erosion has introduced the notion of maximal ball. Its formal definition is the following:

Definition:

- Let us call *ball* of size n and of center x the dilation of the point x for the structuring element n of the family: $\delta_n(x)$
- A ball of size n and center x is *maximal* with respect to the set X, if there exist no other index k and no other center x' such that:

$$\delta_n(x) \subset \delta_k(x') \subset X, \quad k>n$$



If they are symmetrical, the maximal balls identify points which are equidistant from at least two boundary points.

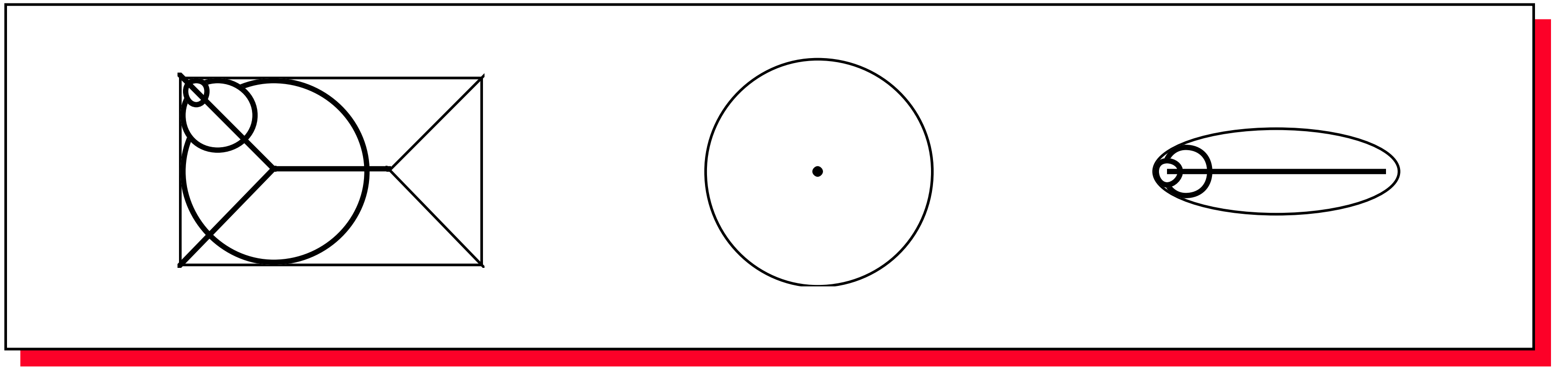
Skeleton: definition

- The ultimate erosion is a set of centers of maximal balls. The set of all centers of maximal balls defines the **skeleton**.

Definition:

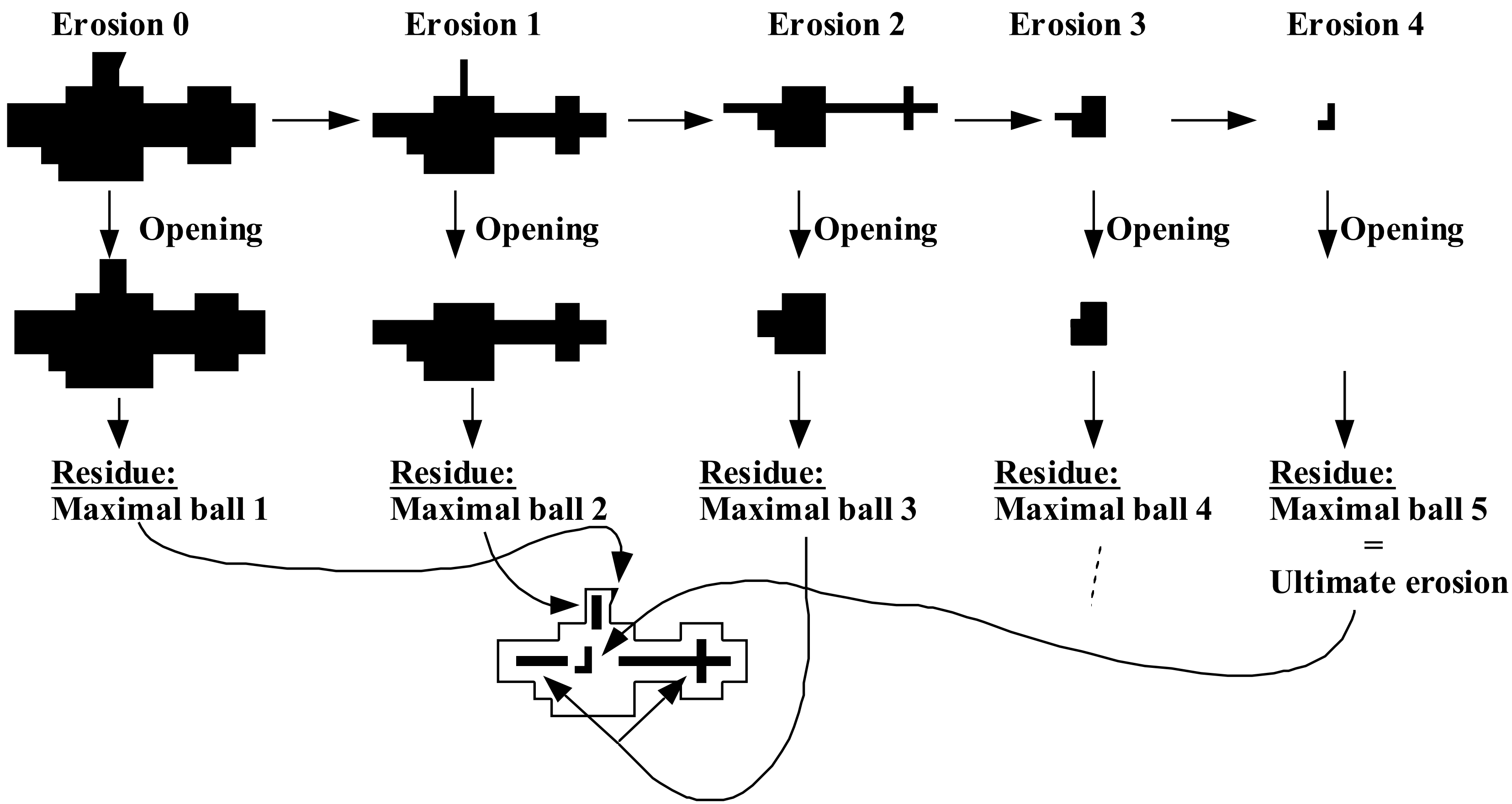
- The skeleton of a set X in the sense of the family $\{\delta_n\}$ is the set of centers x of maximal balls

Note: If the balls are symmetrical, the skeleton represents approximately the medial axis of the set.



Skeleton: Construction

Structuring element: ■



Skeleton: Algorithm

- The algorithm allowing the computation of the skeleton is exactly the same as the one for the ultimate erosion replacing the opening by reconstruction by a unitary opening:

Algorithm:

- In the case of homogeneous family, the skeleton is the residue the families of erosion and of unitary opening of the erosions:
  Residue between $\{\varepsilon_i\}$ and $\{\gamma\varepsilon_i\}$

$$S(X) = \Re_{\{\varepsilon_i\},\ \{\gamma\varepsilon_i\}}(X) = \bigcup_i \Re_{\varepsilon_i,\ \gamma\varepsilon_i}(X) = \bigcup_i S_i(X)$$

- In the case of nonhomogeneous family, the skeleton is defined as:
  Residue between $\{\varepsilon_i\}$ and $\{\gamma_{i+1}\varepsilon_i\}$

Note: The function which has the skeleton as support and whose grey level values are the size of the maximal balls is called the **quench function**.

Skeleton: Properties (I)

Size:

- The skeleton is thin in the sense that its erosion by $\cup \eta_i$ (the elementary erosion in the homogeneous case) is empty.

Anti-extensive and idempotent:

- $\mathbf{X} \supset S(X)$, and
- When the family is homogeneous, $S(S(X)) = S(X)$

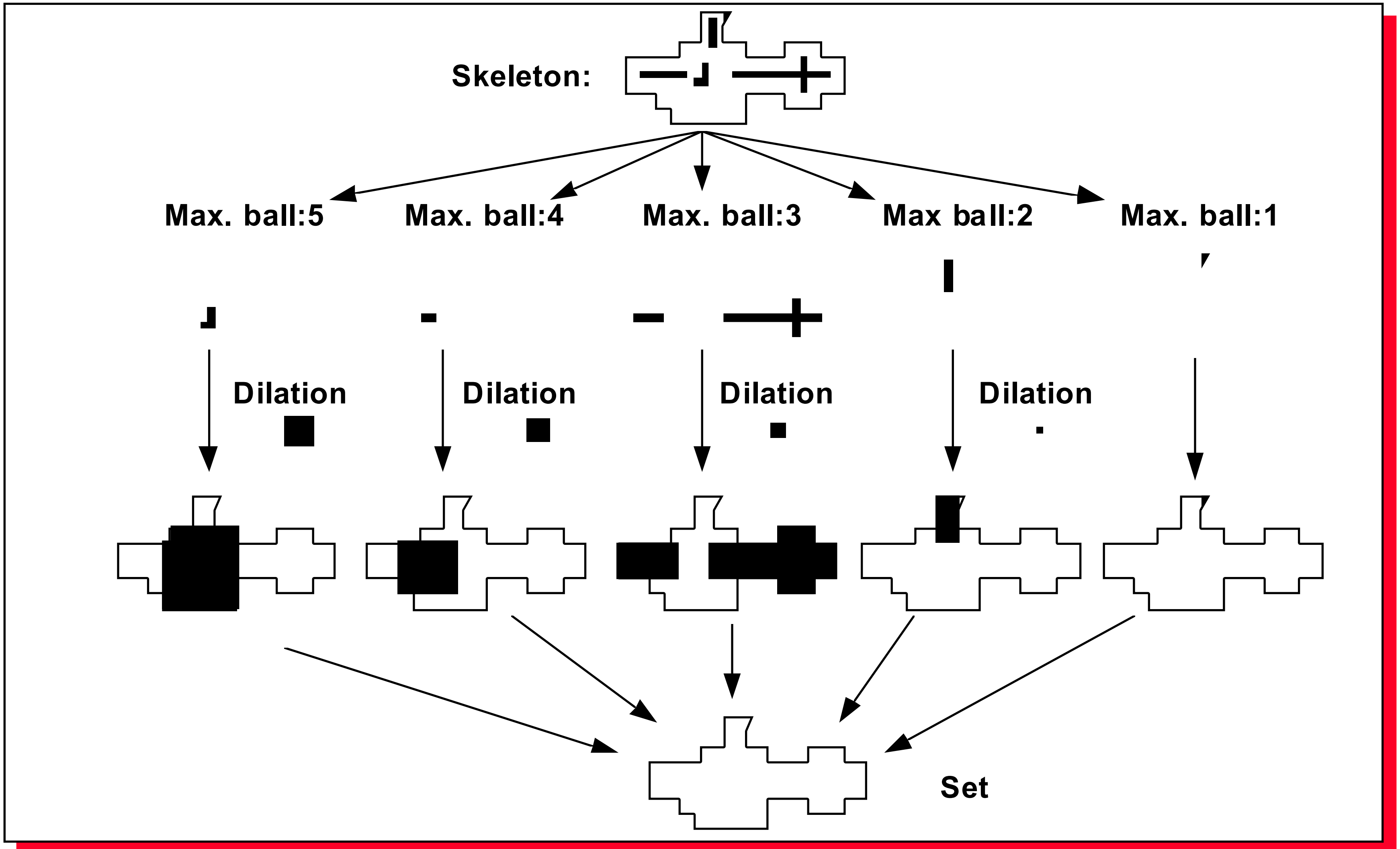
Preservation of the information:

- The set X and its openings can be computed directly from the skeleton and its quench function:

$$X = \bigcup_{i \geq 0} \delta_i(S_i(X)) \qquad \gamma_j(X) = \bigcup_{i \geq j} \delta_i(S_i(X))$$

=> The transformation is invertible and leads to an other representation of the lattice.

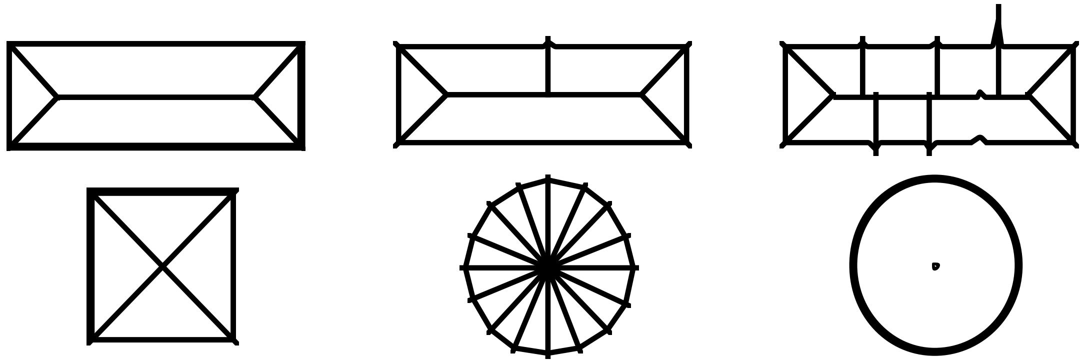
Inversion of the skeleton



Skeleton: Properties(II)

Discontinuity:

- The skeleton transformation is <u>not</u> continuous. Therefore, a small variation of the original set may result in very different skeletons:



=> To solve this problem, smoother versions of the skeleton have been introduced: **Conditional bisector**.

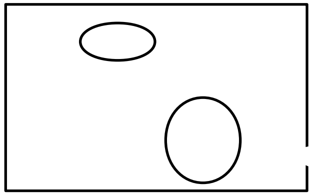
Connectivity:

- Although the connectivity is preserved in the continuous case, this property disappears in the digital case. If the connectivity preservation is of importance, other techniques based on "hit or miss" transforms are used.

Skeleton by Influence Zone: SKIZ

Goal:

- The zone of influence of a component X is the set of points that are closer to X than to any other component.
- The SKIZ the boundary of these zones of influence.

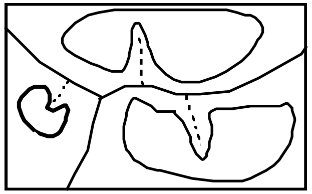


Skiz

Influence zone

Construction:

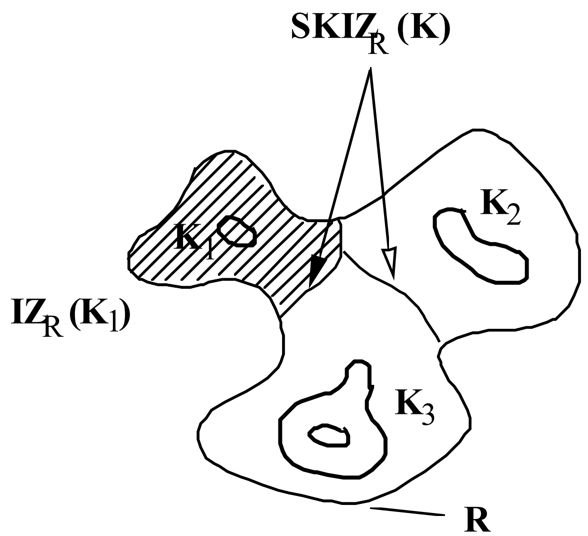
- In the digital case, the SKIZ is constructed in two steps:
  1) "Thinning" of the background (with L in the hexagonal case)
  2) Pruning of the "thinning" result (with E in the hexagonal case)



Geodesic SKIZ

- The geodesic zone of influence of a component K_i in a reference R, $IZ_R(K_i)$, is formed by all points of R which are at a geodesic distance to K_i smaller than to any other component K_j, $j \neq i$.

- The geodesic SKIZ is formed by the boundaries of the the geodesic influence zones of the components inside the reference.

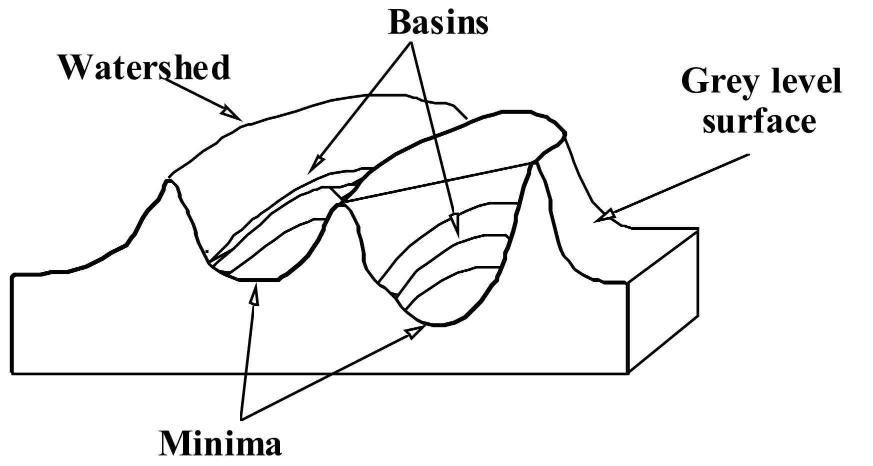


Note: The geodesic distance of a point to a set is the smallest geodesic distances of the point to all points of the set.

Topographic interpretation of the "watershed"

- The name of watershed comes from a topographical analogy where the image is considered as a surface with the grey level values defining the altitude.

- Each local minimum is associated to a basin. If it is raining on the surface, the minimum associated to a basin is the point that would receive the water coming from the point of the basin.

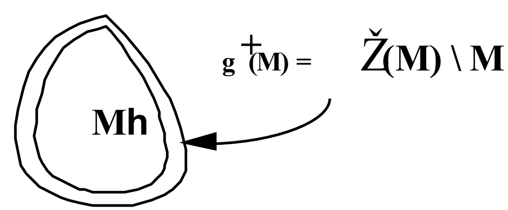


Minimum and associated basin

- Threshold at level h: $T_h(f) = \{ x : f(x)$ Š $h\}$
- The basin points associated to a minimum **M**, **C(M)**, are the surface points that would send water to that minimum. Let us call $\mathbf{C_h (M)}$ these points that are at a level (altitude) lower than h:

$$C_h(M) = \{ x \in C(M) : f(x) \text{ Š } h \} = C(M) \cap T_h(f)$$

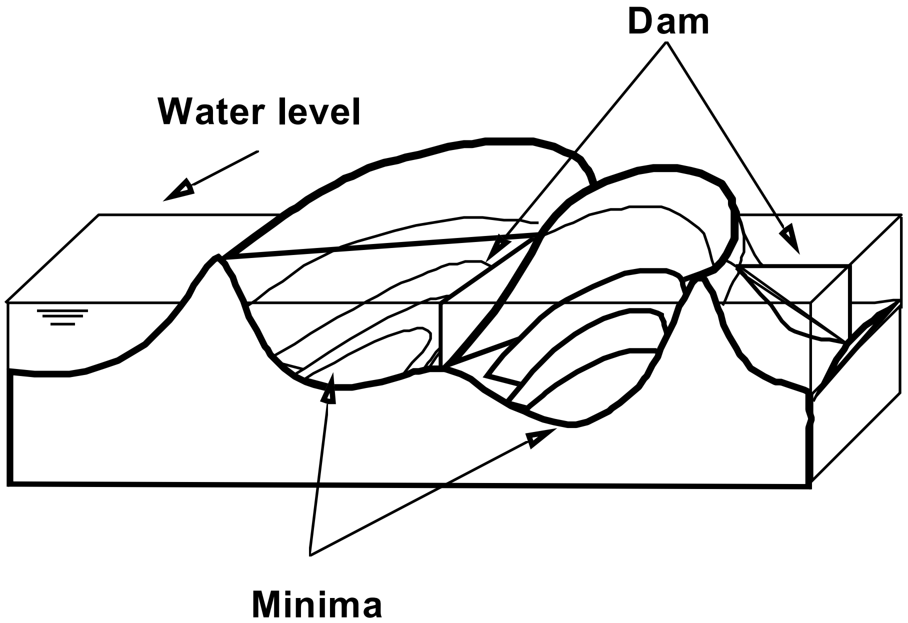
- Minimum of level h, $\mathbf{M_h}$, is a connected component of constant value which has only higher neighbors.



$$\forall \ \mathbf{x} \in \mathbf{g^+(M),\ f(x) > h}$$

- $\mathbf{Min_h (f)}$ is the set of all minima at level h.

Construction of "watershed" by flooding (I)



- Suppose that holes are made in each local minimum and that the surface is flooded from these holes. Progressively, the water level will increase.
- In order to prevent the merging of water coming from two different holes, a dam is built at each contact point.
- At the end, the dams are surrounded with water. They constitute the watersheds.

Construction of "watershed" by flooding (II)

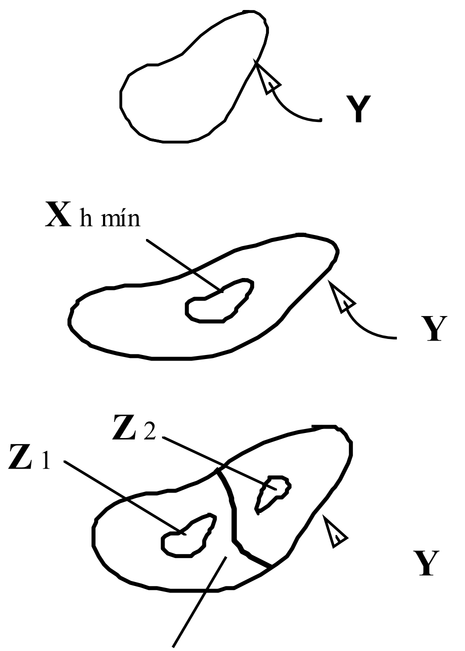
Flooding algorithm:

- Initial set: $X_{h\,min} = T_{h\,min}(f)$
- Threshold at level $h_{min}+1$:

For a connected component Y of $T_{h\,min+1}(f)$ there are three possible inclusions:

1. $Y \cap X = \emptyset$, Y is a new minimum
2. $Y \cap X \neq \emptyset$ and connected, Y are points of the same basin $C_{h\,min+1}(Y \cap X_{h\,min}) = Y$
3. $Y \cap X \neq \emptyset$ and not connected, Y has k different minima $(Z_1, \ldots, Z_k)$. The best choice for the basins are given by their geodesic influence zones:

$$C_{h\,min+1}(Z_i) = IZ_Y(Z_i)$$



SKIZ$_Y$(Z), Z={Z_i}

Construction of "watershed" by flooding (III)

Flooding algorithm (cont):

- Initialization: $X_{h\,min+1} = Min_{h\,min+1}(f) \cup IZ_{T_{hmin+1}(f)}(X_{h\,min})$

Iteration:

1) $X_{h\,min} = T_{h\,min}(f)$

2) $X_{h+1} = Min_{h+1}(f) \cup IZ_{T_{h+1}(f)}(X_h)$

- The set of basins of the image f are given by $X_{h\,max}$
- The set of "watershed" is is complement: $X^C_{h\,max}$

Applications of the "watershed"

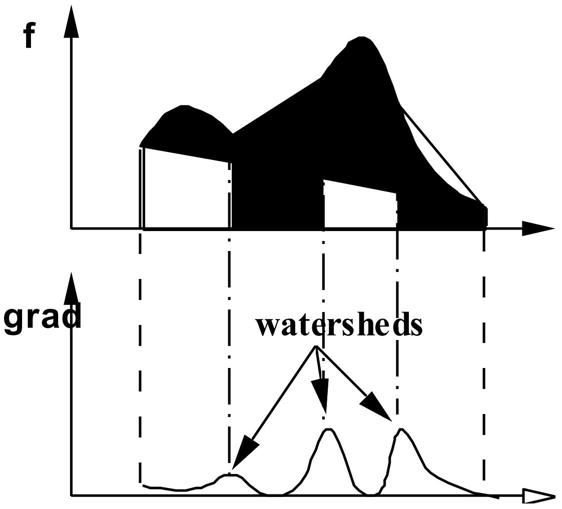
In topography

Study of the draining of a surface by the use of digital models

In signal processing

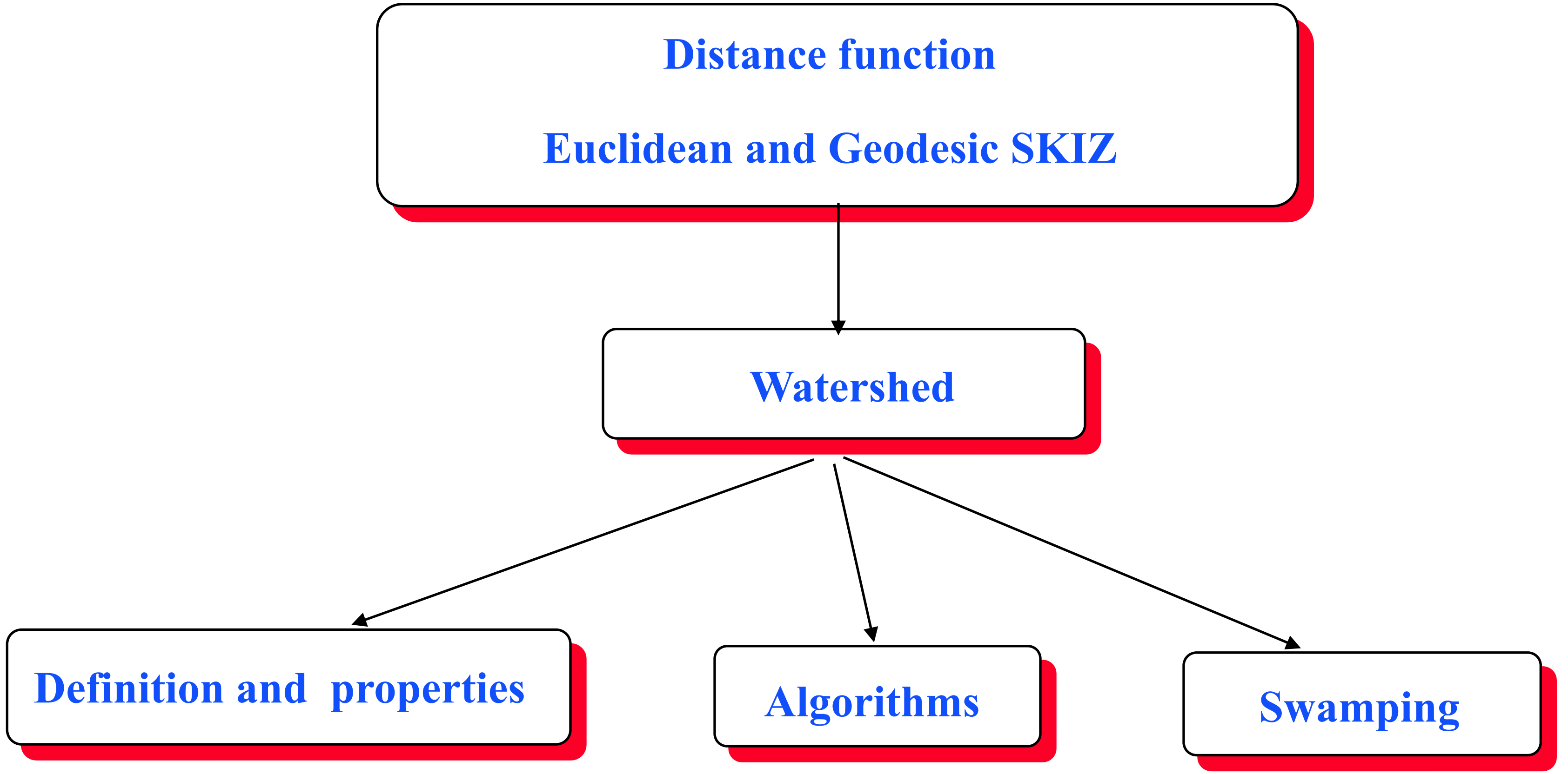
- Contour detection:

  The contour of a signal can be viewed as the "watershed" of its gradient.



- Segmentation

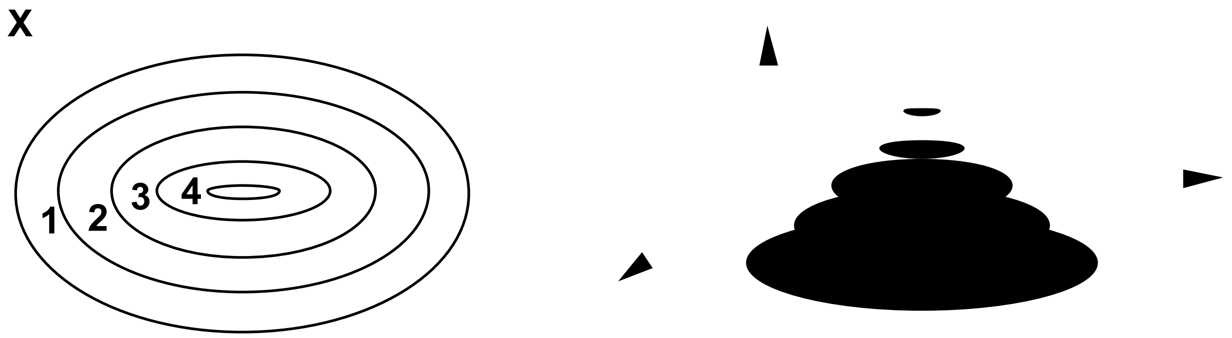
section IX : SKIZ and Watershed



Distance function (I)

Definition:

- The distance function is an intermediate step between sets and functions.
- When a distance has been defined, It is possible to associate with each set X, its subset Xλ composed of all the points which are at a distance larger than λ from its boundary.



- When λ increases, the subsets are included within each other (and parallel in the euclidean case). They can be considered as the horizontal thresholding of a function whose grey level is λ at x if x is at a distance λ from the boundary. This function is called **Distance Function**.

Distance Function (II)

Properties:

- If the distance is characterized by the sets of disks $d\lambda$ of size λ, the subsets $X\lambda$ can be considered as the result of the erosions of X by the disks. More precisely:

  1) $\lambda \geq \mu \quad \Rightarrow \quad \delta_\lambda \geq \delta_\mu$

  2) $\delta_\lambda \delta_\mu \leq \delta_{\lambda+\mu}, \qquad \lambda, \mu \geq 0$

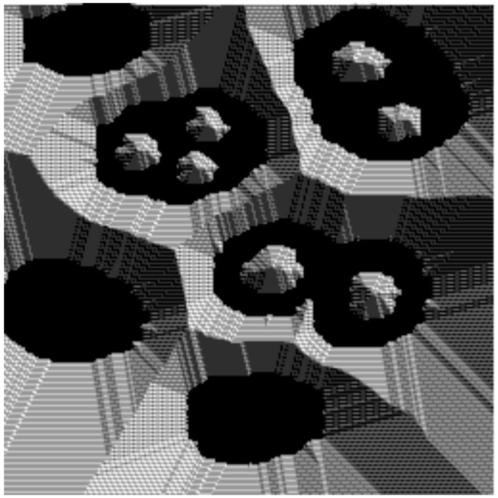
  3) $I = \wedge \{\delta_\lambda, \ \lambda > 0\}, \qquad$ I:Identity

- Conversely, each family of symmetrical dilations which fulfills these three rules **generates a distance** d which is characterized by:

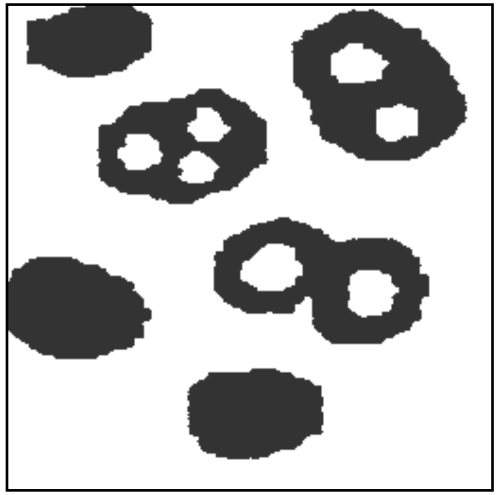
$$d(x,y) = \mathrm{Inf}\,\{x \in \delta_\lambda(y)\} = \mathrm{Inf}\,\{y \in \delta_\lambda(x)\}$$

where $\delta_\lambda(y)$ is the disk of center y and radius λ.

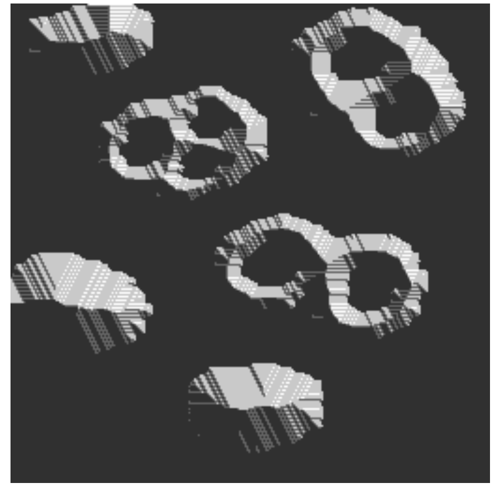
Distance function (III): an Example



Distance to X



Set X



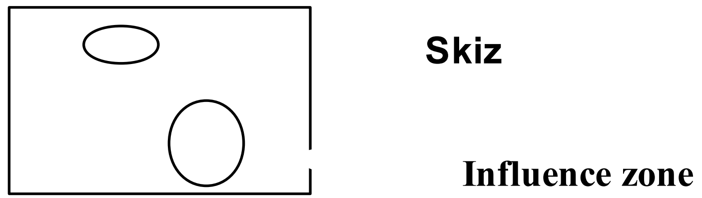
Distance to X^c

In the close loops of crest lines, the pixels are equidistant to two (or more) holes, in the left cae, and to two (or more) particles in the right case.They delineate the **zones of influence** of the objects. This nice property suggests to study them specifically.

SKIZ, or Skeleton by Influence Zone

Goal:

- In a metric space, the zone of influence of a component X is the set of points that are closer to X than to any other component.
- The **SKIZ** the boundary of these zones of influence.



Construction:

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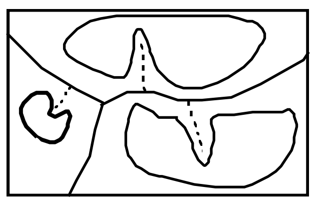


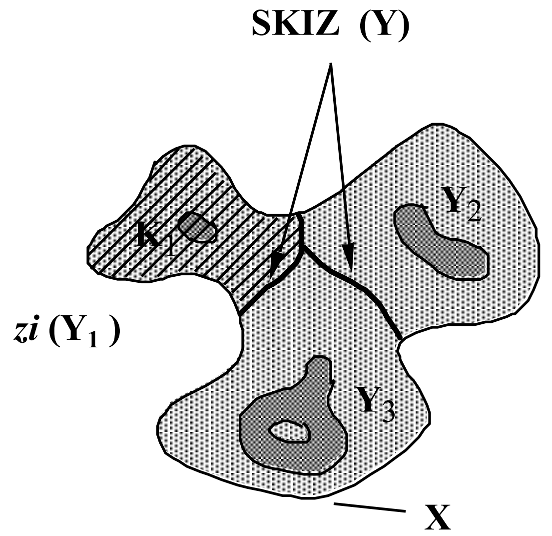
Geodesic SKIZ

- Let $Y = \cup\{ Y_i , i \in I \}$ be a set of I compact connected components included in a mask X .
- The geodesic zone of influence of a component Y_i in X, is formed by all points of X whose geodesic distance to Y_i is smaller than to any other component of Y *i.e.*

$$zi\,(Y_i \setminus X) = \{a \in X ,\ \forall k \neq i,\ d_X(a, Y_j) \leq d_X(a, Y_k)\}$$

where the geodesic distance from point a to set Y is the inf of the geodesic distances from a to all points of Y.

- The geodesic SKIZ is then the boundaries between the geodesic zones of influence.



Example of geodesic SKIZ

The Two Problems of Segmentation (I)

- When one wants to segment a set, the first question which arises is :
  " into how many pieces ? " (in case of figure 1, into 6 or 7 particles?)
- One can decide and indicate, manually, the supposed locations of the centers.
- Alternatively, one can trust in a marking technique. However, the results risk to vary with the method (here, between 6 and 7).
- In all cases, this first step is a **choice**.

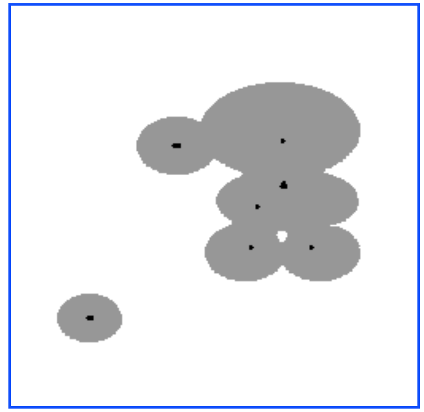
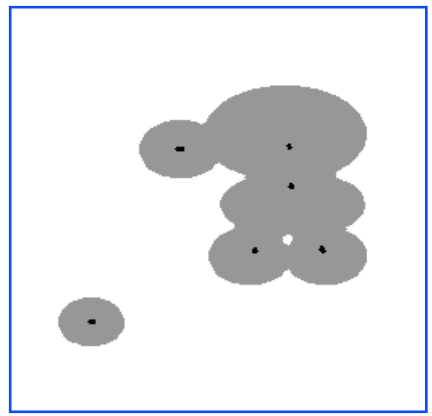


Figure 1 :

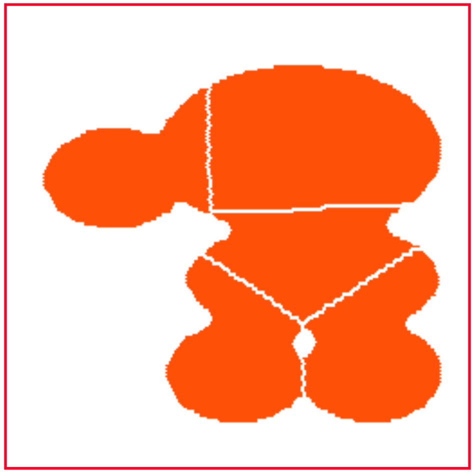
A set and its ultimate erosion



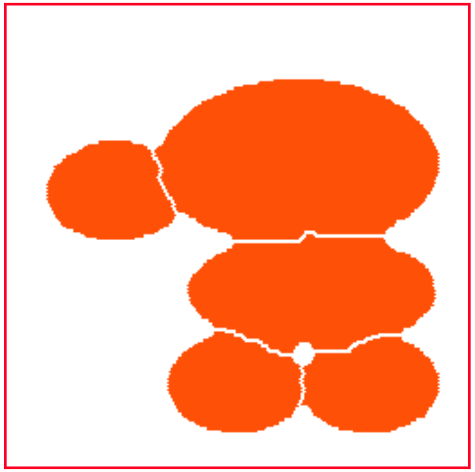
Ultimate erosion after filtering of the set.

The Two Problems of Segmentation (II)

- Given a certain choice of markers, (here, the conditional bisector) le segmentation lines may be **optimised :**
  - A coarse expression is obtained by taking the exoskeleton of the markers (the shape of the set is then just ignored)
  - One can partly take this shape into account by dilating each marker by a disc equal to the difference between number of steps for the ultimate marker and that for the current one;
  - Following this idea, the finest procedure consists in calculating the geodesic skiz of eroded n° i inside eroded n° i-1, as i varies from the ultimate eroded to zero, and taking the union of the resulting skiz's.



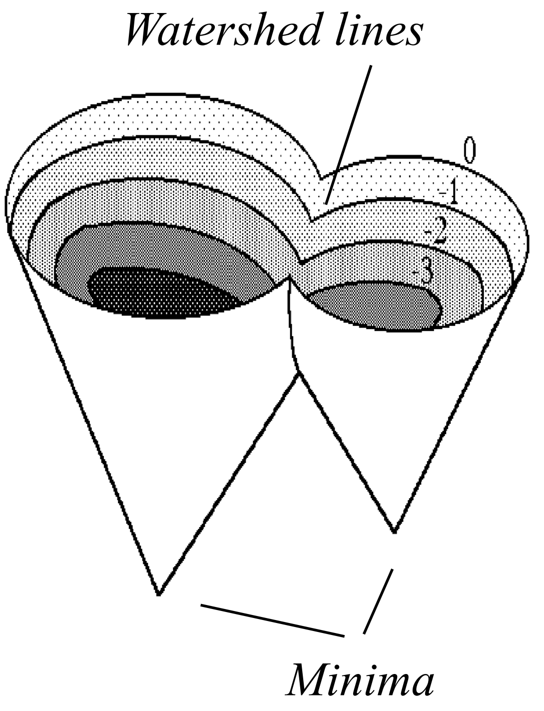
Exoskeleton of the markers.



Successive geodesic skiz's.

Fine Segmentation fine Distance Function

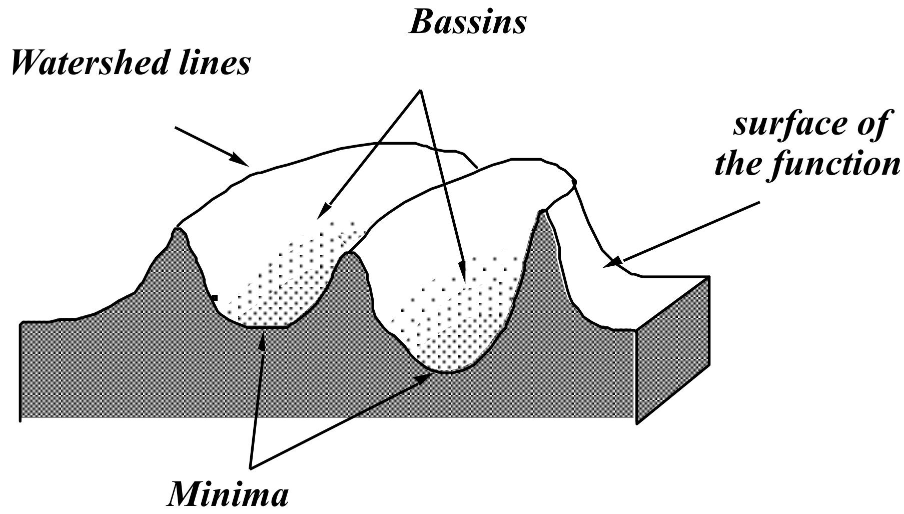
- The successive eroded versions of a set X are nothing but the horizontal sections of its distance function ;
- therefore the finer previous segmentation, by means of geodesic skiz's, comes back to build up the **watershed lines** of this distance function (at least when the les markers are the ultimate erosions).
- By duality, they also appear to be the **valleys lines** on the inverse function.



Watershed Lines for Numerical Functions

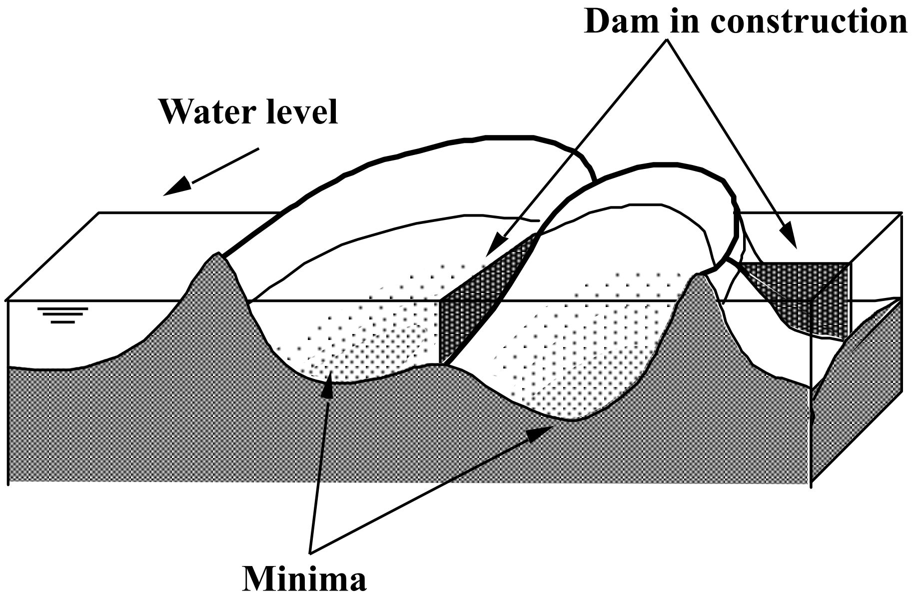
- Indeed, the method developed for the set case via the distance functions applies as well to any **numerical image**. The analogy between gray levels and altitudes still justifies the topographical terms of **bassins** and **watersheds**.

- However, it is less matter of rain water running down to the minima than, on the opposite, of water which **springs from the minima** .



topographical patterns of a numerical image

Construction of the Watersheds by Flooding (I)



- Suppose that holes are made in each local minimum and that the surface is flooded from these holes. Progressively, the water level will increase.
- In order to prevent the merging of water coming from two different holes, a dam is built at each contact point.
- At the end, the dams are surrounded with water. They constitute the **watersheds**.

Construction of the Watersheds by Flooding (II)

Flooding algorithm :

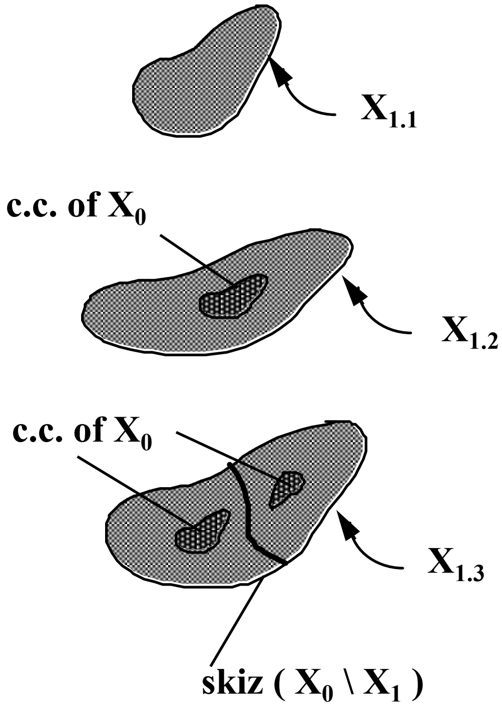
- Let m be the minimum value of function f. Put:

$$X_0 = \{ x: f(x) = m \},$$

and $X_k = \{ x: f(x) \le m+k \}$ with $1 \le k \le \max f$

- Denote by Y_1 the geodesic zones of influence of X_0 inside X_1. Three types of connected components of X_1 have to be distinguished :
  - thoses, $X_{1,1}$ that do not contain points of X_0: then they do not belong to Y_1
  - thoses, $X_{1,2}$ that contain a unique c.c. of X_0 : then they fully belong to Y_1
  - thoses, $X_{1,3}$ that contain several c.c. of X_0 : Y_1 recovers then $X_{1,3}$ minus the branches of its geodesic skiz.

evolution of the flooding



Construction of the Watersheds by Flooding (III)

- Since the $X_{1,1}$'s are minima which appear at level 1, we have to incorporate them to the flooding proccess. Thus we replace

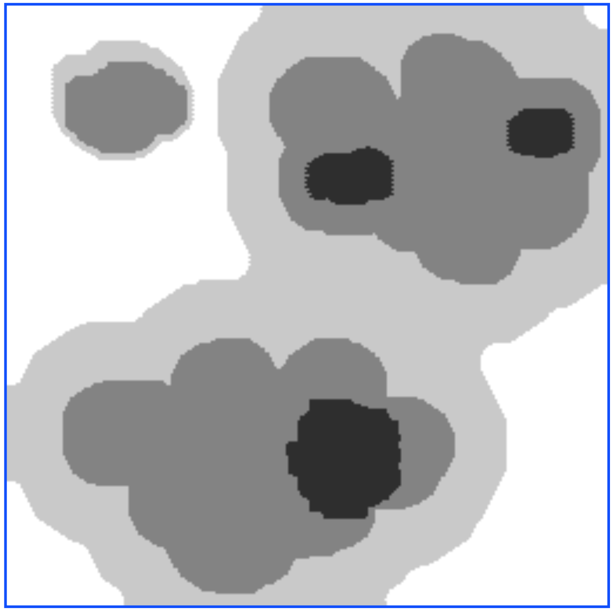
$$X_1 \quad \text{by} \quad Y_1 \cup X_{1,1}$$

- ...and we iterate. The geodesic zones of influence
  - Y_2 of $Y_1 \cup X_{1,1}$ inside X_2 are clculated ;
  - They provide markers $Y_2 \cup X_{2,1}$; etc...
- The proccess ends when level k = max f is reached. Then one has :

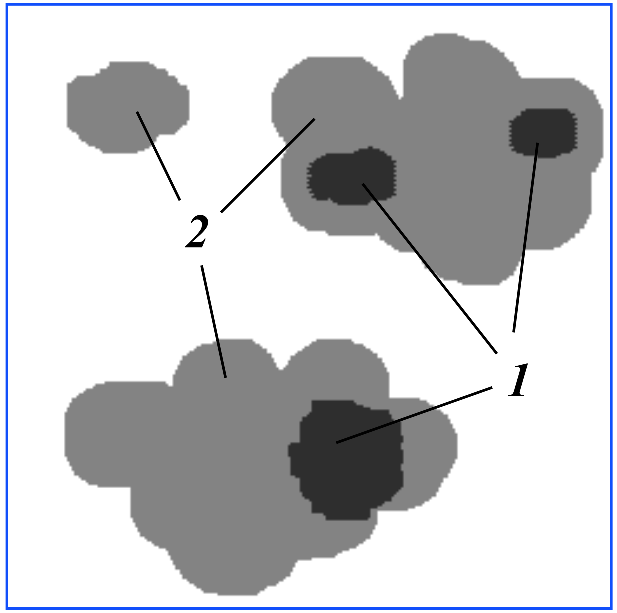
$$Y_{\max f} = \text{union of the bassins;}$$

$$[\,Y_{\max f}\,]^c = \text{Watershed lines.}$$

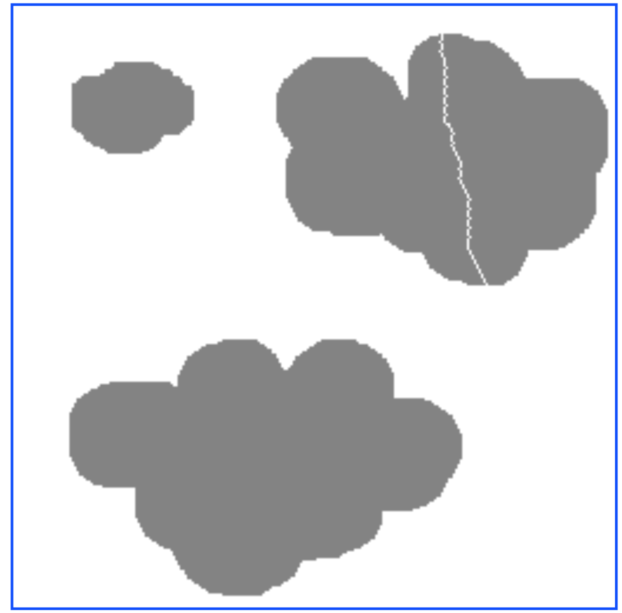
Example of Watershed by Flooding (I)



Initial image.

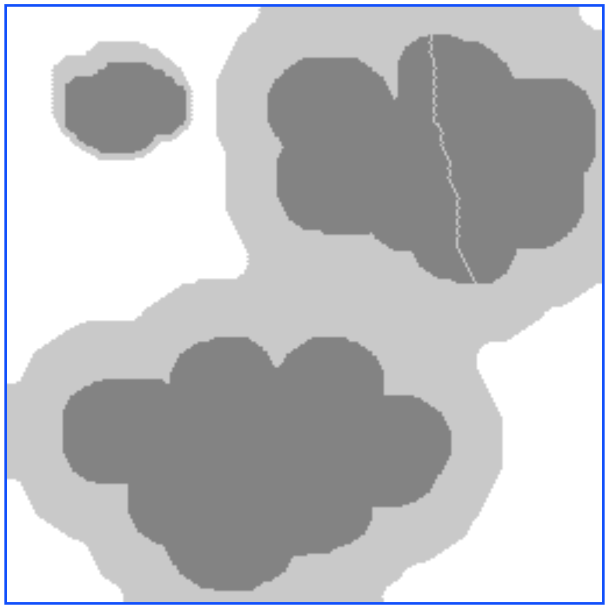


Minima (1), and next level (2).

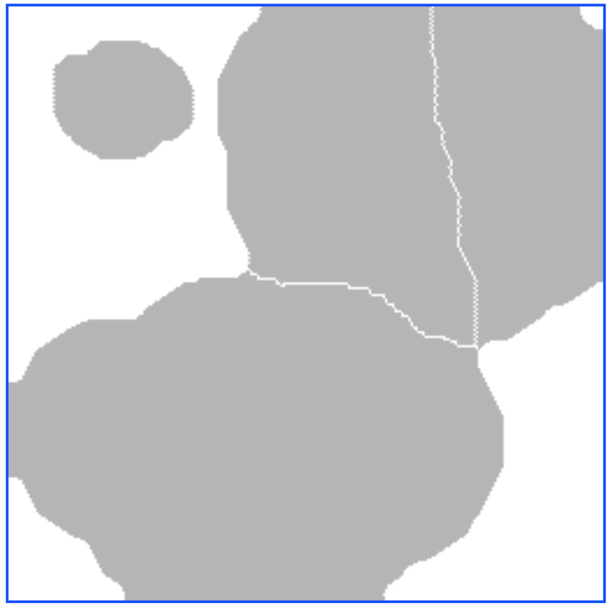


Geodesic skiz of (1) into (2) (in white lines).

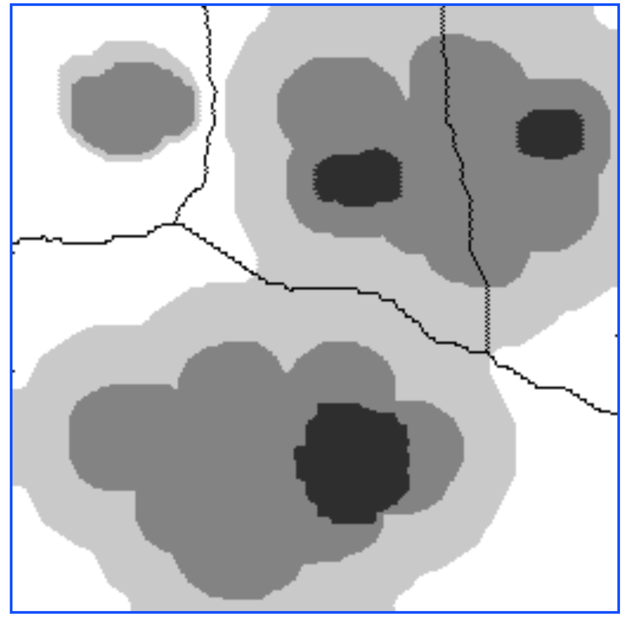
Example of Watershed by Flooding (II)



Level 2, minus the first skiz , and level 3.



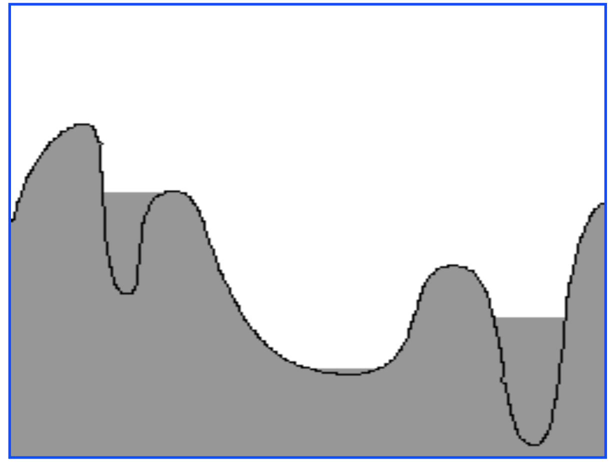
Second skiz (note that it prolongates the first one).



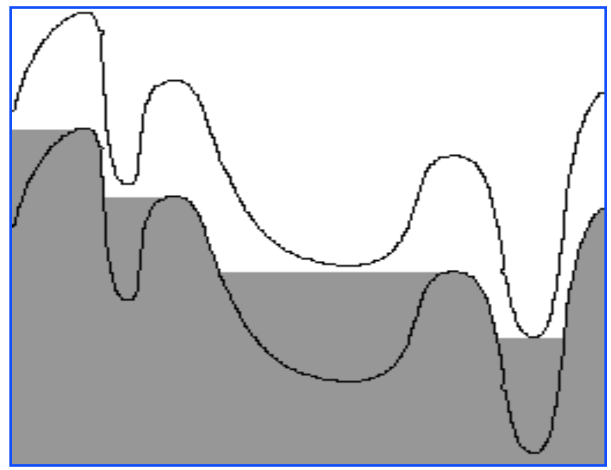
Final watershed (The result is significant in spite of the small number of gray levels).

Minima selection by Filtering

- As a general rule, images have **too many minima**, and a careless computation of theirs watersheds often leads to a disastrous **over-segmentation**.
- In order to obtain significant minima, one can begin with filtering the images:
  - either "***horizontally***" by plane alternating filters, with or without reconstruction ;
  - or "***vertically***" by closings $\varphi^{rec}(f;f+h)$ of dynamic h. In particular, for h =1 all the minima are extracted.
- When dealing with **maxima**, one takes $\gamma^{rec}(f;f-h)$.

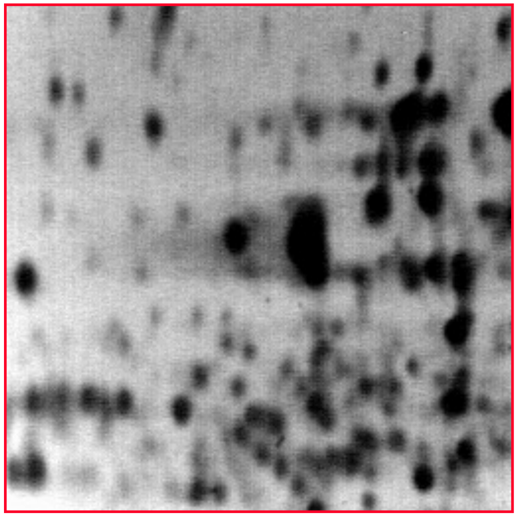


" horizontal" filtering

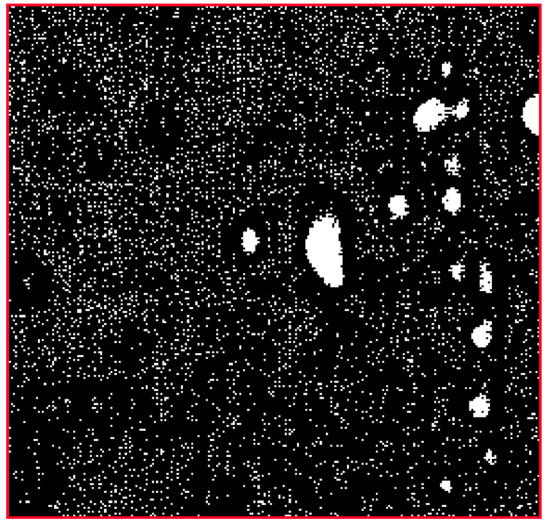


"vertical" filtering

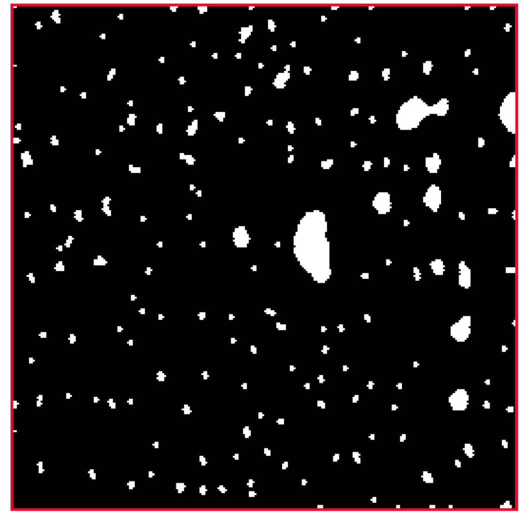
Example of Minima Filtering



(a) initial image : electrophoresis gel.



(b) minima of the initial image.



(c) minima of the image after alternating filtering of (a) by the unit hexagon.

N.B. *There is no viual difference between (a) and its filtered version .*

Changing the Minima : Swamping

- The markers may not coincide with the minima of f. In that case, they act on f via the **swamping** transformation.
- **Goal :** Given $f \geq 0$, and a set M of markers, find the inf of the functions
  - whose minima are exclusively the connected components of M;
  - which are zero on M and $\geq f$ on M^c .
- **Way:** Associate with M function g such that $g(x) = 0$ if $x \in M$; $g(x) = \max$ if $x \notin M$. Then, function $\varphi^{rec}(f;g)$ provides the required inf . It is called Swamping of f by M.