Algorithm design and analysis

— Greedy algorithms —

Silvio Guimarães

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Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2023

Algorithm Design

- Start discussion of different ways of designing algorithms.
- Greedy algorithms, divide and conquer, dynamic programming.
- Discuss principles that can solve a variety of problem types.
- Design an algorithm, prove its correctness, analyse its complexity.

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- Design an algorithm, prove its correctness, analyse its complexity.
- Greedy algorithms: make the **current best choice**.

Algorithm design and analysis

— Coin change —

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Coin change

INSTANCE Let C be a set of coins $\{c_1, c_2, \cdots, c_n\}$ in which c_i means a coin of a specific value and $c_i = c_j$ if $i = j$. Let S be the amount of the change.

SOLUTION The smallest number of coins to achieve the amount S.

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What's the smallest number of coins to achieve $S = 8$?

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- $C = \{1, 2, 6\}$ and $S = 8$

What's the smallest number of coins to achieve $S = 8$? 2 coins.

Design an algorithm to compute the smallest number of coins.

Algorithm design and analysis

— Interval Scheduling —

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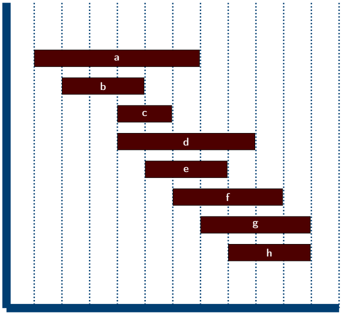
Interval Scheduling

INSTANCE Nonempty set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION The largest subset of mutually compatible jobs.



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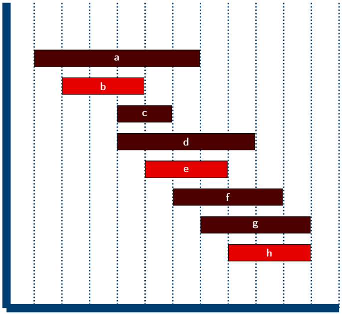

- Two jobs are compatible if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.

Template for Greedy Algorithm

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
- Key question: in what **order** should we process the jobs?

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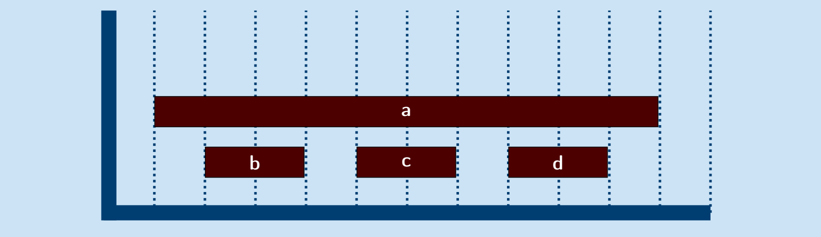

The best solution has **3** compatible jobs. But the it depends on the order in which the jobs are processed !!!!

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
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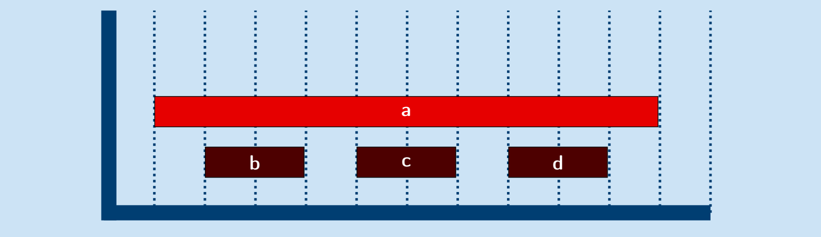
Earliest start time – Increasing order of start time $s(i)$.



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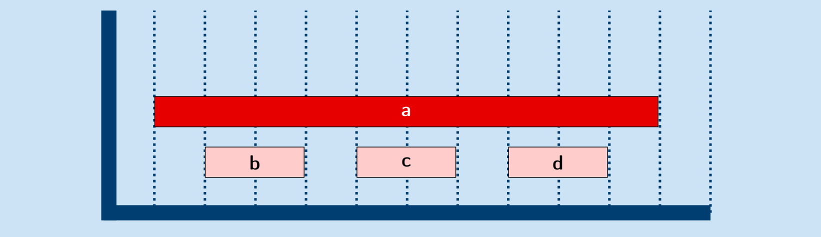

The number of compatible jobs using this strategy is **1**, against **3** jobs in the best solution!!!

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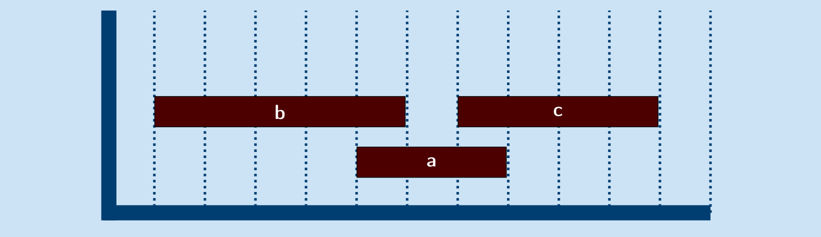

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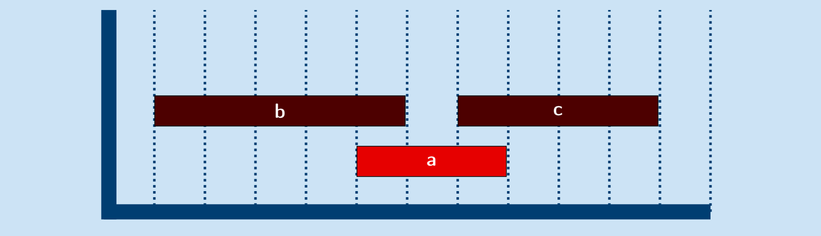
Shortest interval – Increasing order of length $f(i) - s(i)$.



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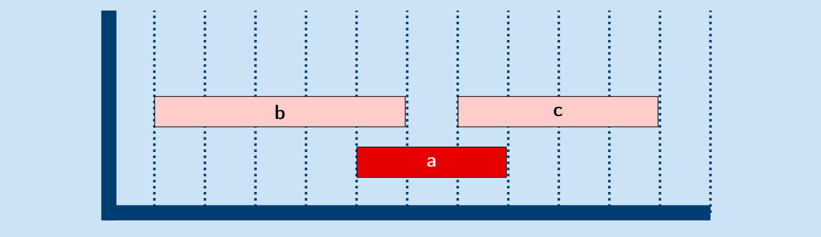

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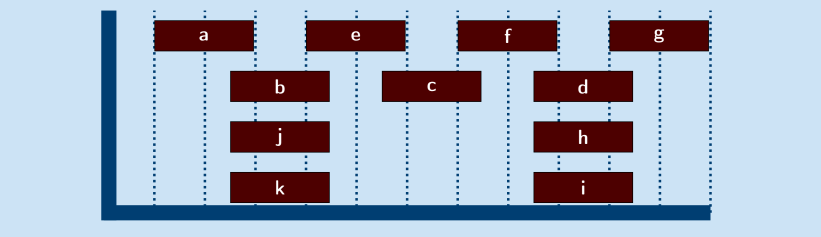

The best solution has **4** compatible jobs. But the it depends on the order in which the jobs are processed !!!!

- Process jobs in some order. Add next job to the result if it is compatible with the jobs already in the result.
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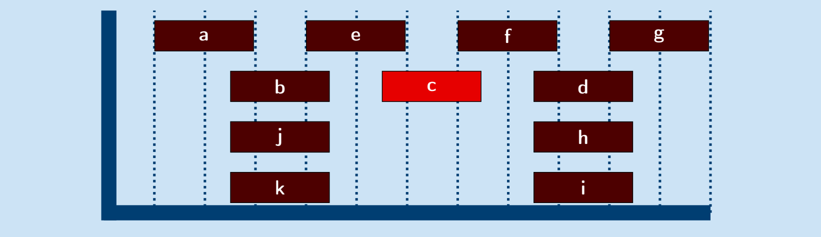
Fewest conflicts – Increasing order of the number of conflicting jobs



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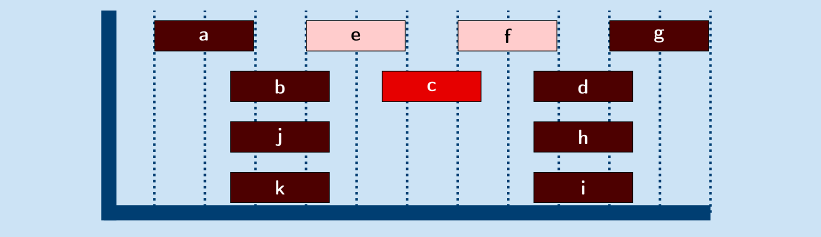

The number of compatible jobs using this strategy is **3**, against **4** jobs in the best solution!!!

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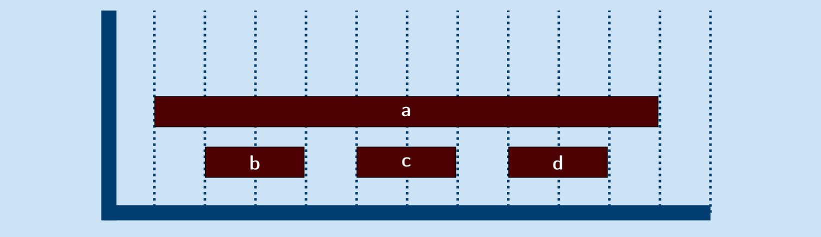

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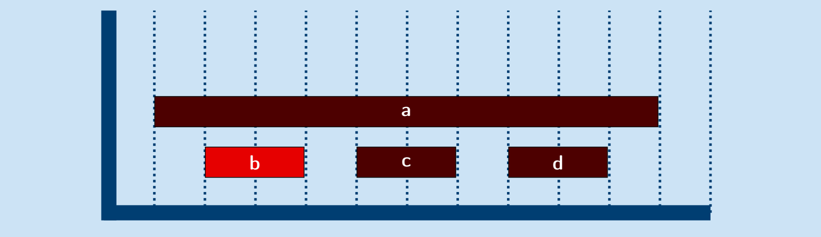

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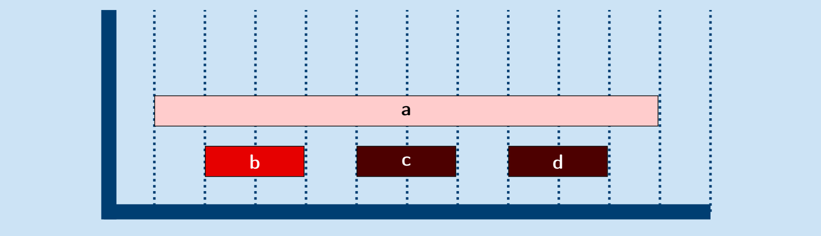

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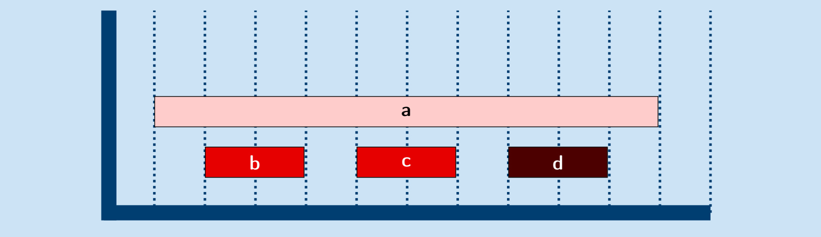

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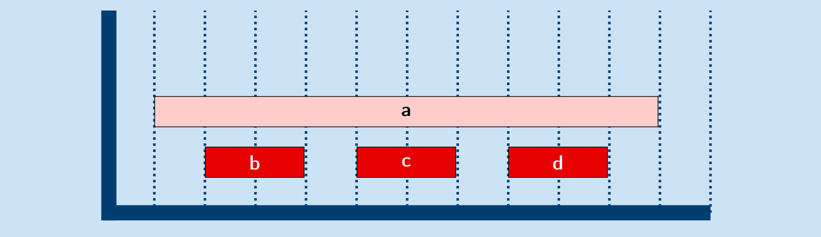

The number of compatible jobs using this strategy is **3**.

IS Algorithm: Earliest Finish Time (EFT)

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input : A set of jobs R
output: A set of compatible jobs A

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Let R be the set of all jobs;
Let A be an empty set for representing the solution;
while R is not empty do
    Choose a job i ∈ R that has the smallest finishing time;
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Analysing the EFT Algorithm

- Let O be an optimal set of jobs. We will show that $|A| = |O|$.
- Let $i_1, i_2, \ldots, i_k$ be the set of jobs in A in order.
- Let $j_1, j_2, \ldots, j_m$ be the set of jobs in O in order.
- Claim: For all indices $r \leq k$, $f(i_r) \leq f(j_r)$. Prove by induction on r.

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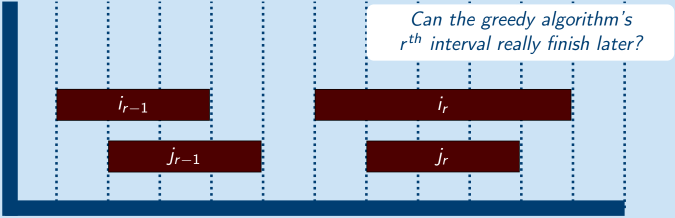

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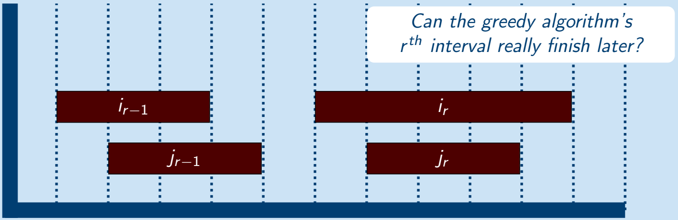

- Claim: The greedy algorithm returns an optimal set A.

Implementing the EFT Algorithm

Reorder jobs so that they are in **increasing order of finish time**.

Store starting time of jobs in an array S.

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Running time is $O(n \log n)$, **dominated by sorting**.

Algorithm design and analysis

— Interval Partitioning —

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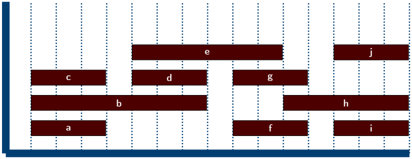
Interval Partitioning

INSTANCE Set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION A partition of the jobs into k sets, where each set of jobs is mutually compatible, and k is minimised.

This schedule uses 4 classrooms to schedule 10 lectures.



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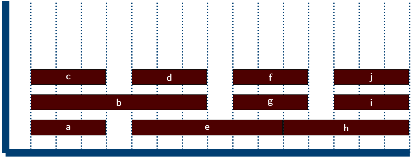

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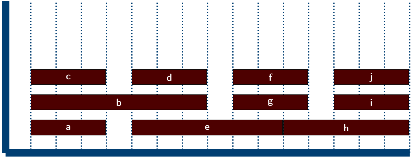
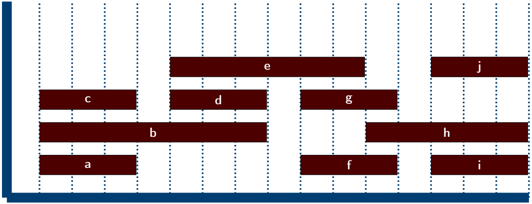
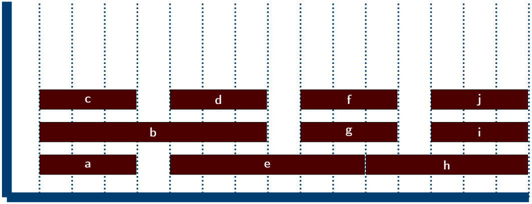


- This problem models the situation where you have set of fixed jobs, and you want to schedule all jobs using as few resources as possible.

Depth of Intervals





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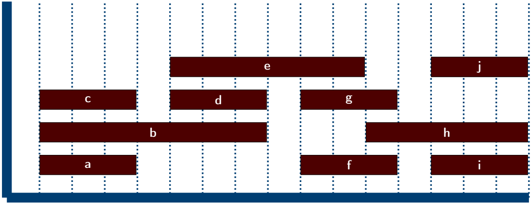

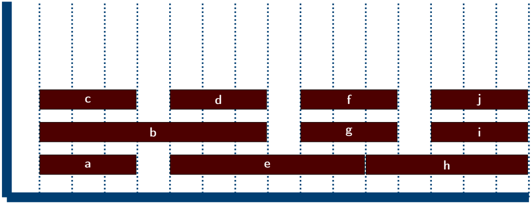

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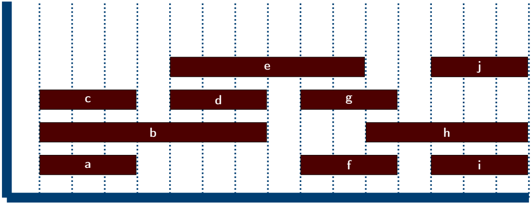

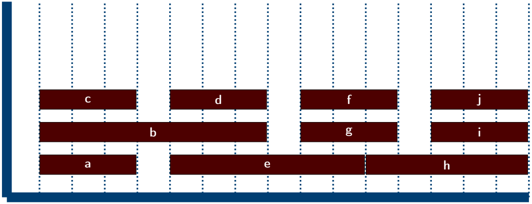

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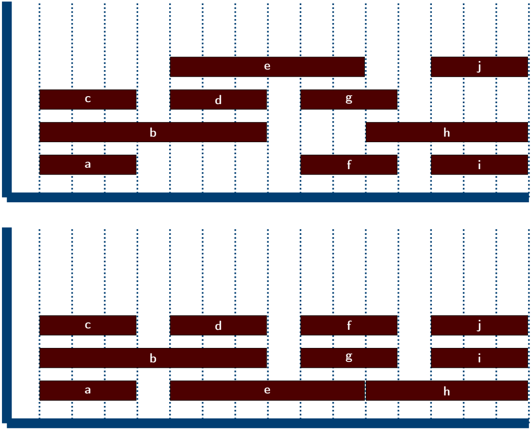

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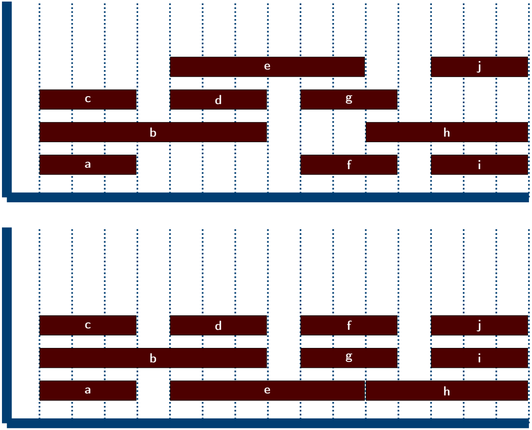

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- Is it possible to compute k efficiently? Is $k =$ depth?

Interval Partitioning Algorithm

Algorithm: Interval partitioning algorithm

input : A set of jobs R
output: K sets of mutually compatible jobs

Sort the interval by their start times, breaking ties arbitrarily;
Let $I_1, I_2, \cdots, I_n$, denote the interval in this order;

for $j = 1$ **to** n **do**
  foreach *interval I_i that preceds I_j in sorted order and overlaps it* **do**
    Exclude the labels of I_i from consideration for I_j
  end
  if *there is any label from $\{1, 2, \cdots, d\}$ that has not been excluded* **then**
    Assign a nonexcluded label to I_j
  else
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- Every interval gets a label and **no pair of overlapping intervals** get the same label.
- The greedy algorithm is **optimal**.

Interval Partitioning Algorithm

Algorithm: Interval partitioning algorithm

input : A set of jobs R

output: K sets of mutually compatible jobs

```
Sort the interval by their start times, breaking ties arbitrarily;
Let I_1, I_2, ..., I_n, denote the interval in this order;
for j = 1 to n do
    foreach interval I_i that preceds I_j in sorted order and overlaps it do
        Exclude the labels of I_i from consideration for I_j
    end
    if there is any label from {1, 2, ..., d} that has not been excluded then
        Assign a nonexcluded label to I_j
    else
        Leave I_j unlabeled
    end
end
```

- Every interval gets a label and **no pair of overlapping intervals** get the same label.
- The greedy algorithm is **optimal**.
- The running time of the algorithm is $O(n \log n)$.

Algorithm design and analysis

— Minimising Lateness —

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Scheduling to Minimise Lateness

- Study different model: job i has a length $t(i)$ and a deadline $d(i)$.
- We want to schedule all jobs on one resource.
- Our goal is to assign a starting time $s(i)$ to each job such that each job is delayed as little as possible.
- A job i is **delayed** if $f(i) > d(i)$; the **lateness of the job** is $\max(0, f(i) - d(i))$.
- The **lateness of a schedule** is $\max_i \max(0, f(i) - d(i))$.

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t_i	3	2	1	4	3	2
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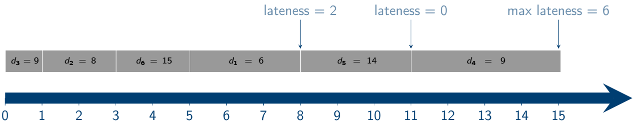

Template for Greedy Algorithm

Minimise Lateness

INSTANCE Set $\{(t(i), d(i)), 1 \leq i \leq n\}$ of lengths and deadlines of n jobs.

SOLUTION Set $\{s(i), 1 \leq i \leq n\}$ of start times such that $\max_i \max(0, s(i) + t(i) - d(i))$ is as small as possible.

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- Key question: In what order should we schedule the jobs?
  Shortest length *Increasing order of length $t(i)$.*
  Shortest slack time *Increasing order of $d(i) - t(i)$.*
  Earliest deadline *Increasing order of deadline $d(i)$.*

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Increasing order of length $t(i)$.

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counter-example

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Shortest slack time

Increasing order of $d(i) - t(i)$.

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counter-example

Minimising Lateness: Earliest Deadline First (EDF)

Algorithm: Minimising lateness algorithm

input : A set of jobs R

output: The set of scheduled interval $[s(i), f(i)]$ for $i = 1, \cdots, n$

Sort the jobs in order of their deadlines;

Assume, for simplicity, that $d_1 \leq \cdots \leq d_n$;

Initially, $f = s$;

for $j = 1$ **to** n **do**

  Assign the job i to the time interval from $s(i) = f$ to $f(i) = f + t_i$;

  Let $f = f + t_i$

end

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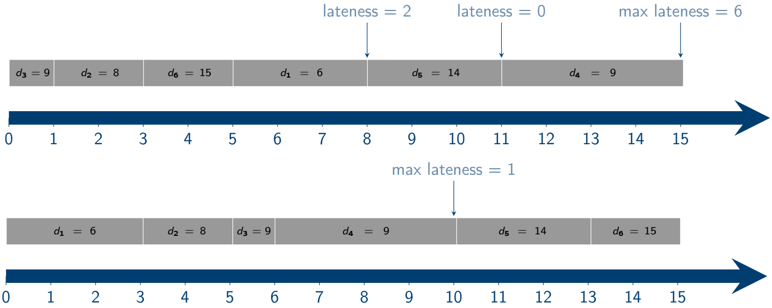

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```

- Proof of correctness is more complex.
- We will use an exchange argument: gradually modify the optimal schedule O till it is the same as the schedule A computed by the algorithm.

Properties of Schedules

- A schedule has an **inversion** if a job j with deadline $d(j)$ is scheduled before a job i with an earlier deadline $d(i)$, i.e., $d(i) < d(j)$ and $s(j) < s(i)$.

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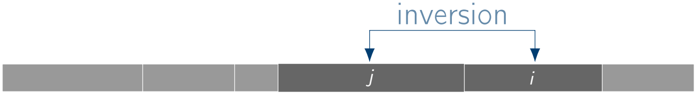

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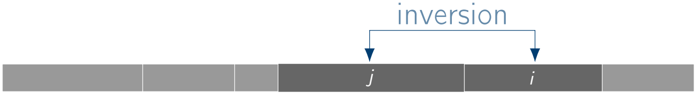

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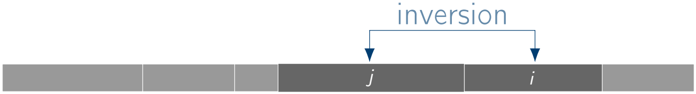

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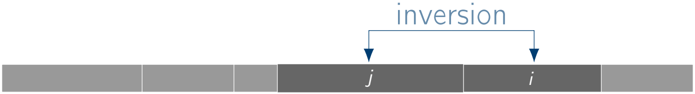

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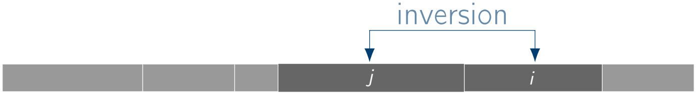

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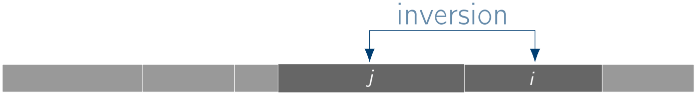

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- The greedy algorithm produces an **optimal schedule**.

Properties of the Optimal Schedule

- Claim: the optimal schedule O has no inversions and no idle time.
  1. If O has an inversion, then there is a pair of jobs i and j such that i is scheduled just before i and $d(i) < d(j)$.

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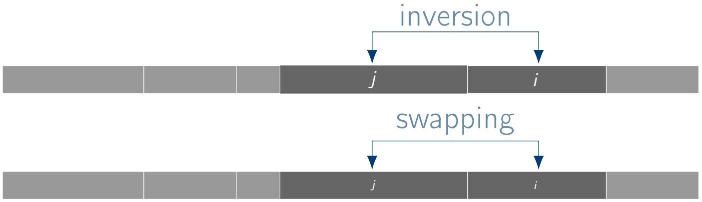
  2. Let i and j be consecutive inverted jobs in O. After swapping i and j, we get a schedule O' with one less inversion.
  3. The maximum lateness of O' is no larger than the maximum lateness of O.
- If we can prove the last item, we are done, since after $\binom{n}{2}$ swaps, we obtain a schedule with no inversions whose maximum lateness is no larger than that of O.

Swapping Inverted Jobs



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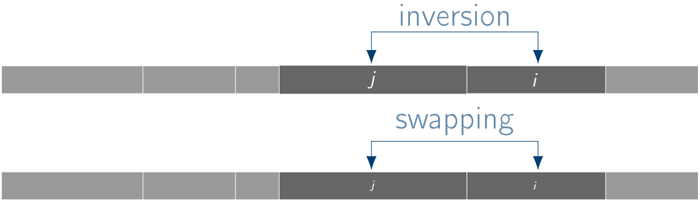

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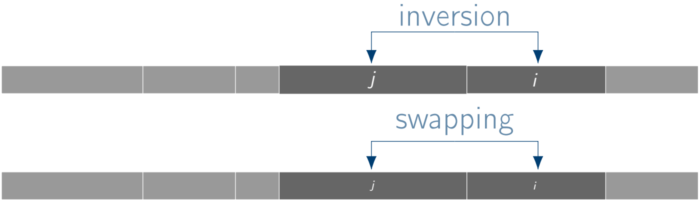

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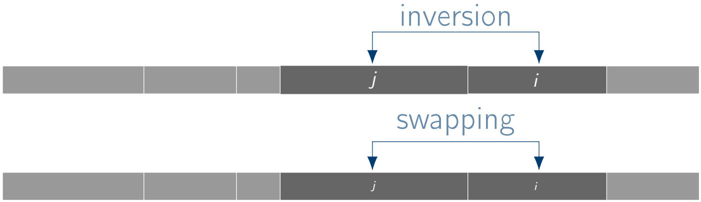

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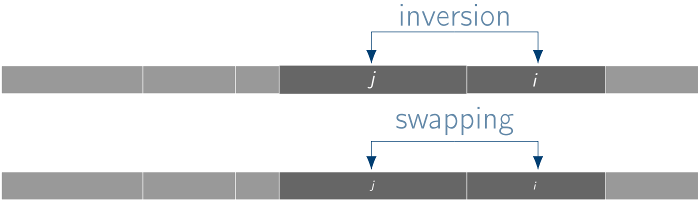

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- $l'(j) \leq l(j)$.
- $l'(i) \leq l(j)$.

Summary

- Greedy algorithms make local decisions.
- Three analysis strategies:

  Greedy algorithm stays ahead *Show that After each step in the greedy algorithm, its solution is at least as good as that produced by any other algorithm.*

  Structural bound *First, discover a property that must be satisfied by every possible solution. Then show that the (greedy) algorithm produces a solution with this property.*

  Exchange argument *Transform the optimal solution in steps into the solution by the greedy algorithm without worsening the quality of the optimal solution.*