Algorithm design and analysis

— Divide and conquer —

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— Mergesort —

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Divide and Conquer

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- Solve each part recursively.
- Solve base cases by brute force.
- Efficiently combine solutions for sub-problems into final solution.
- Common use:
  - Partition problem into two equal sub-problems of size $n/2$.
  - Solve each part recursively.
  - Combine the two solutions in $O(n)$ time.
  - Resulting running time is $O(n \log n)$.

Mergesort

SORT

INSTANCE Nonempty list $L = x_1, x_2, \ldots, x_n$ of integers.

SOLUTION A permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that $y_i \leq y_{i+1}$, for all $1 \leq i < n$.

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- Mergesort is a divide-and-conquer algorithm for sorting.
  1. Partition L into two lists A and B of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively.
  2. Recursively sort A.
  3. Recursively sort B.
  4. Merge the sorted lists A and B into a single sorted list.

Merging Two Sorted Lists

Algorithm: Intercalation

input : $A = a_1, a_2, \ldots, a_k$ and $B = b_1, b_2, \ldots b_l$.
output: The distances of the vertices from s

Maintain a current pointer for each list;
Initialise each pointer to the front of the list;
while *both lists are nonempty* **do**
    Let a_i and b_j be the elements pointed to by the *current* pointers;
    Append the smaller of the two to the output list;
    Advance the current pointer in the list that the smaller element belonged to;
end
Append the rest of the non-empty list to the output.

1	4	8	9

3	5	6	7

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```

Running time of this algorithm is $\boxed{O(k+l)}$.

Analysing Mergesort

- Worst-case running time for n elements ($T(n)$) is at most the sum of the worst-case running time for $\lfloor n/2 \rfloor$ elements, for $\lceil n/2 \rceil$ elements, for splitting the input into two lists, and for merging two sorted lists.
- Assume n is a power of 2.

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- Three basic ways of solving this recurrence relation:
  1. "Unroll" the recurrence (somewhat informal method).
  2. Guess a solution and substitute into recurrence to check.
  3. Guess solution in $O()$ form and substitute into recurrence to determine the constants.

Unrolling the recurrence

- Recursion tree has log n levels.
- Total work done at each level is cn.
- Running time of the algorithm is $cn \log n$.

Substituting a Solution into the Recurrence

- Guess that the solution is $cn \log n$ (logarithm to the base 2).
- Use induction to check if the solution satisfies the recurrence relation.
- Base case: $n = 2$. Is $T(2) = c \leq 2c \log 2$? Yes.
- Inductive step: assume $T(m) \leq cm \log_2 m$ for all $m < n$. Therefore, $T(n/2) \leq (cn/2) \log n - cn/2$.

$$
\begin{aligned}
T(n) &\leq 2T(n/2) + cn \\
&\leq 2\left((cn/2) \log n - cn/2\right) + cn \\
&= cn \log n
\end{aligned}
$$

Partial Substitution

- Guess that the solution is $kn \log n$ (logarithm to the base 2).
- Substitute guess into the recurrence relation to check what value of k will satisfy the recurrence relation.
- $k \geq c$ will work.

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- Divide into q sub-problems of size $n/2$ and merge in $O(n)$ time. Two distinct cases: $q = 1$ and $q > 2$.
- Divide into two sub-problems of size $n/2$ and merge in $O(n^2)$ time.

Algorithm design and analysis

— Counting inversions —

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Divide and Conquer Algorithms

- Study three divide and conquer algorithms:
  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.

Divide and Conquer Algorithms

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  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.
- Third problem uses a clever divide strategy.

Motivation

- Collaborative filtering match one user's preferences to those of other users.

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- Suggestion: two rankings are very similar if they have few inversions.

Motivation

- Collaborative filtering match one user's preferences to those of other users.
- Meta-search engines merge results of multiple search engines to into a better search result.
- Fundamental question: how do we compare a pair of rankings?
- Suggestion: two rankings are very similar if they have few inversions.
  - Assume one ranking is the ordered list of integers from 1 to n.
  - The other ranking is a permutation $a_1, a_2, \ldots, a_n$ of the integers from 1 to n.
  - The second ranking has an inversion if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.

Counting Inversions

INSTANCE A list $L = x_1, x_2, \ldots, x_n$ of distinct integers between 1 and n.

SOLUTION The number of pairs $(i, j), 1 \leq i < j \leq n$ such $a_i > a_j$.

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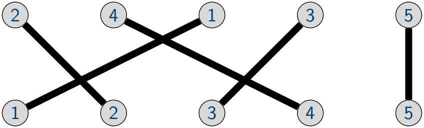

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1 5 4 8 10 2 6 9 12 11 3 7

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$5-4, 5-2, 4-2, 8-2, 10-2$

$6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7$

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Counting Inversions: Algorithm

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- Candidate algorithm:
  1. Partition L into two lists A and B of size $n/2$ each.
  2. Recursively count the number of inversions in A.
  3. Recursively count the number of inversions in B. and one element in B.

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Key idea: problem is much easier if A and B are sorted!

Counting Inversions: Conquer Step

Counting Inversions: Final Algorithm

Algorithm: Sort and count

input : The list L of elements
output: The number of inversion and the sorted list L

```
if |L| = 1 then
    there is no inversions;
else
    Divide the list into two halves: A and B;
    (r_A, A) = sort-and-count(A);
    (r_B, B) = sort-and-count(B);
    (r, L) = merge-and-count(A, B);
end
r = r_A + r_B + r
```

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  $(r_A, A) = \text{sort-and-count}(A)$;
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  $(r, L) = \text{merge-and-count}(A, B)$;
end
$r = r_A + r_B + r$

Running time $T(n)$ of the algorithm is $O(n \log n)$ because

$$T(n) \leq 2T(n/2) + O(n).$$

Algorithm design and analysis

— Some exercises —

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Finding element

Let A be an array with n numbers. Design a divide-and-conquer algorithm for finding the position of the largest element in the array A.

11	12	7	4	8	5	9	3

Finding element

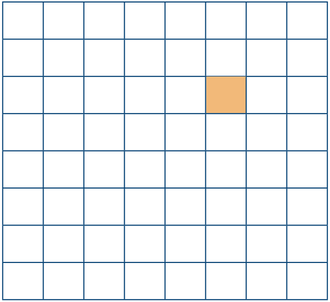
Let A be an array with n numbers. Design a divide-and-conquer algorithm for finding both the smallest and largest elements in the array A.

11	12	7	4	8	5	9	3

Tromino puzzle

Tromino puzzle

A tromino is an L-shaped tile formed by adjacent 1-by-1 squares. The problem is to cover any 2^n-by-2^n chessboard with one missing square (anywhere on the board) with trominoes. Trominoes should cover all the squares of the board except the missing one with no overlaps.



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Integer Multiplication

MULTIPLY INTEGERS

INSTANCE Two n-digit binary integers x and y

SOLUTION The product xy

Multiply Integers

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- Multiply two n-digit integers.

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- Result has at most $\boxed{2n}$ digits.

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SOLUTION The product xy

- Multiply two n-digit integers.
- Result has at most $2n$ digits.
- Algorithm we learnt in school takes

$$
\begin{array}{r}
12 \\
\times\ 13 \\
\hline
36 \\
12 \\
\hline
156
\end{array}
\qquad\qquad
\begin{array}{r}
1100 \\
\times\ 1101 \\
\hline
1100 \\
0000 \\
1100 \\
1100 \\
\hline
10011100
\end{array}
$$

Integer Multiplication

MULTIPLY INTEGERS

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SOLUTION The product xy

- Multiply two n-digit integers.
- Result has at most $2n$ digits.
- Algorithm we learnt in school takes $O(n^2)$ operations. Size of the input is not 2 but $2n$

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- Assume integers are binary.
- Let us use divide and conquer

Divide-and-Conquer Algorithm

- Assume integers are binary.
- Let us use divide and conquer by splitting each number into first $n/2$ bits and last $n/2$ bits.
- Let x be split into x_0 (lower-order bits) and x_1 (higher-order bits) and y into y_0 (lower-order bits) and y_1 (higher-order bits).

$$xy =$$

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$$\begin{aligned} xy &= (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0) \\ &= x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0. \end{aligned}$$

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- Each of x_1, x_0, y_1, y_0 has $n/2$ bits, so we can compute $x_1 y_1$, $x_1 y_0$, $x_0 y_1$, and $x_0 y_0$ recursively, and merge the answers in $O(n)$ time.

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$$T(n) \leq 4T(n/2) + cn$$

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- Assume integers are binary.
- Let us use divide and conquer by splitting each number into first $n/2$ bits and last $n/2$ bits.
- Let x be split into x_0 (lower-order bits) and x_1 (higher-order bits) and y into y_0 (lower-order bits) and y_1 (higher-order bits).

$$\begin{aligned} xy &= (x_1 2^{n/2} + x_0)(y_1 2^{n/2} + y_0) \\ &= x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0. \end{aligned}$$

- Each of x_1, x_0, y_1, y_0 has $n/2$ bits, so we can compute $x_1 y_1$, $x_1 y_0$, $x_0 y_1$, and $x_0 y_0$ recursively, and merge the answers in $O(n)$ time.
- What is the running time $T(n)$?

$$\begin{aligned} T(n) &\leq 4T(n/2) + cn \\ &\leq O(n^2) \end{aligned}$$

Improving the Algorithm

- Four sub-problems lead to an $O(n^2)$ algorithm.

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Final Algorithm

Algorithm design and analysis

— Closest Pair of Points —

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Feb 2023

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
- Started in 1975 by Shamos and Hoey.
- Problems studied have applications in a vast number of fields: ecology, molecular biology, statistics, computational finance, computer graphics, computer vision, . . .

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Closest Pair of Points

INSTANCE A set P of n points in the plane

SOLUTION The pair of points in P that are the closest to each other.

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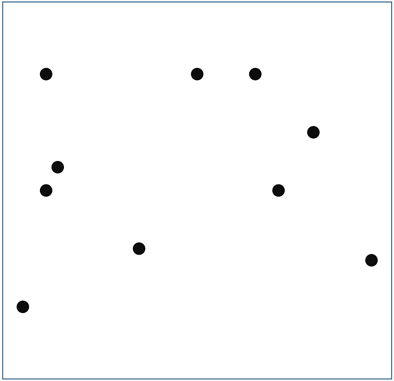
- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.



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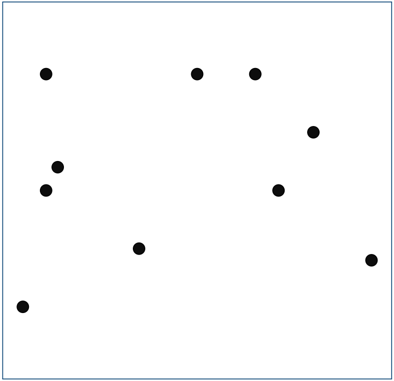

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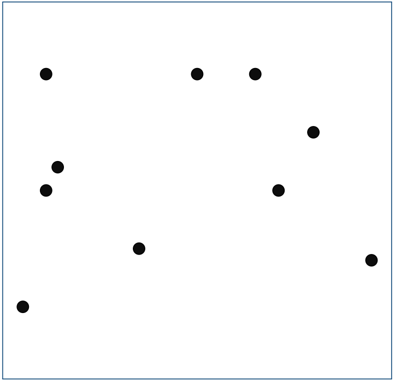

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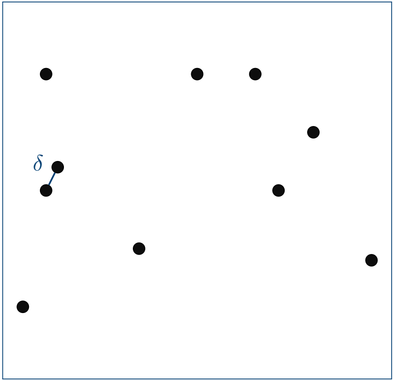

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Assignment

Implement the problem to find the closest pair in a plane.