Programa de Pós-graduação em
INFORMÁTICA

PUC Minas

IM
SCIENCE

Algorithm design and analysis

— Dynamic programming —

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

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Build up a solution incrementally, myopically optimizing some local criterion.

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    - Pro: simple to develop algorithm skeleton.
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    - Con: conquer step can be very hard to implement efficiently.
4. Dynamic programming
    - More powerful than greedy and divide-and-conquer strategies.
    - Implicitly explore space of all possible solutions.
    - Solve multiple sub-problems and build up correct solutions to larger and larger sub-problems.
    - Careful analysis needed to ensure number of sub-problems solved is polynomial in the size of the input.

History of Dynamic Programming

- Bellman pioneered the systematic study of dynamic programming in the 1950s.

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- Bellman pioneered the systematic study of dynamic programming in the 1950s.
- Dynamic programming = “planning over time.”
- The Secretary of Defense at that time was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - “it’s impossible to use dynamic in a pejorative sense”
  - “something not even a Congressman could object to” Reference:
  - Bellman, R. E., *Eye of the Hurricane, An Autobiography*.

Applications of Dynamic Programming

- Computational biology: Smith-Waterman algorithm for sequence alignment.
- Operations research: Bellman-Ford algorithm for shortest path routing in networks.
- Control theory: Viterbi algorithm for hidden Markov models.
- Computer science (theory, graphics, AI, . . .): Unix `diff` command for comparing two files.

Algorithm design and analysis

— Weighted interval scheduling —

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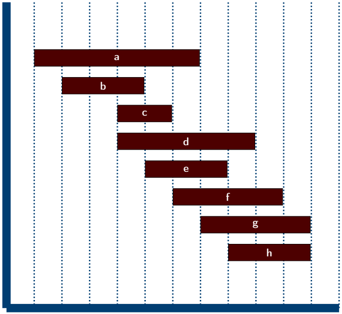
Review: Interval Scheduling

INTERVAL SCHEDULING

INSTANCE Nonempty set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION The largest subset of mutually compatible jobs.



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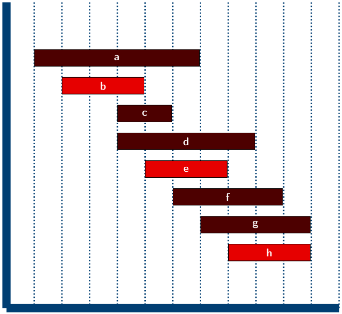


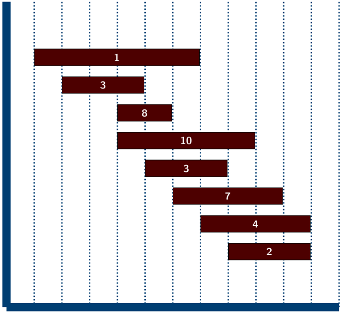
- Two jobs are compatible if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- Greedy algorithm sort jobs in non decreasing order of finish times. Add next job to current subset only if it is compatible with previously-selected jobs.

Weighted Interval Scheduling

WEIGHTED INTERVAL SCHEDULING

INSTANCE Nonempty set $\{(s_i, f_i), 1 \leq i \leq n\}$ of start and finish times of n jobs and a weight $v_i \geq 0$ associated with each job.

SOLUTION A set S of mutually compatible jobs such that $\sum_{i \in S} v_i$ is maximised.



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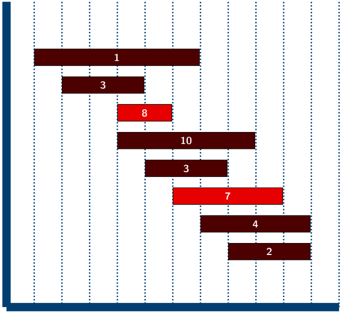
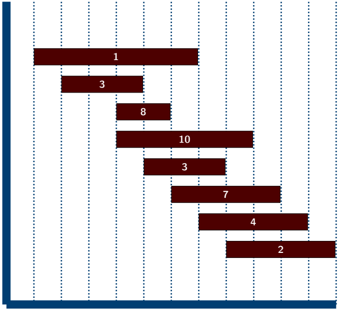


- Two jobs are compatible if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many weighted jobs as possible.
- Greedy algorithm can produce arbitrarily bad results for this problem.

Approach

- Sort jobs in increasing order of finish time and relabel: $f_1 \leq f_2 \leq \ldots \leq f_n$.
- Request i comes before request j if $i < j$.



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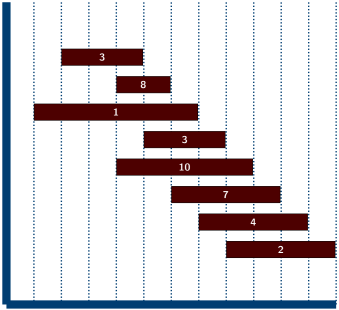

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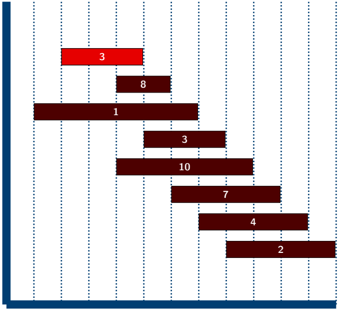

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- We will develop optimal algorithm from very obvious statements about the problem.

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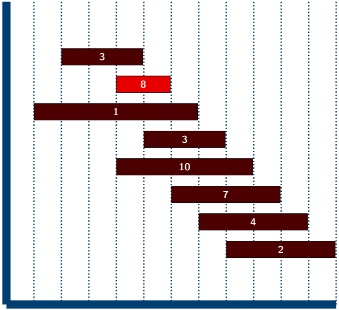

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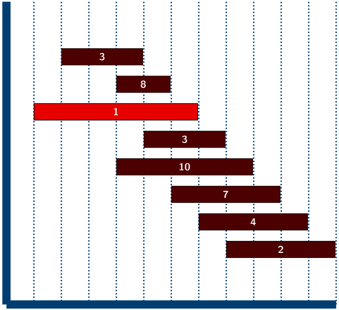

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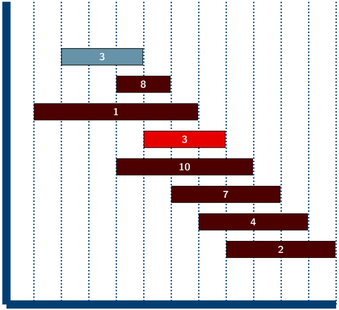

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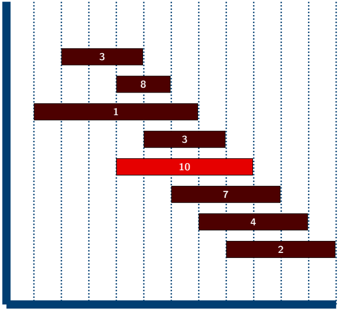

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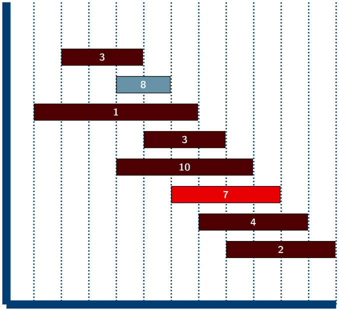

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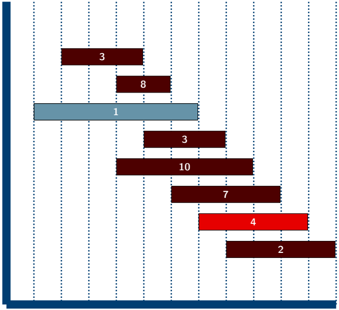

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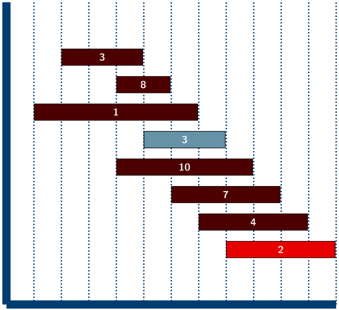

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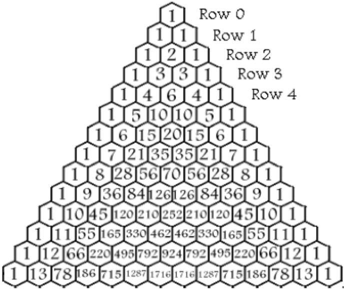
Detour: a Binomial Identity



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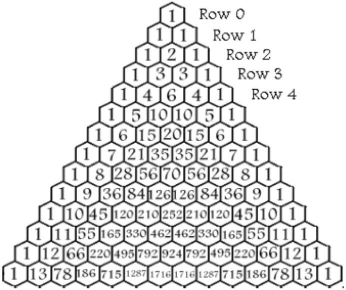

- Pascal's triangle:
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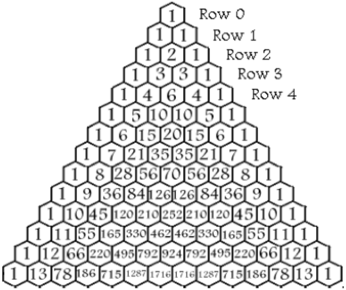

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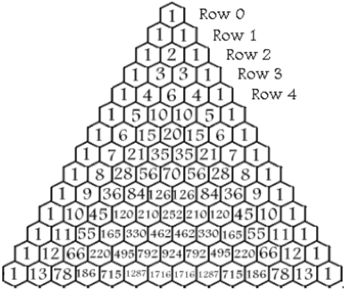

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- Proof: either we select the *n*th element or not . . .

Sub-problems

- Let $\mathcal{O}$ be the optimal solution. Two cases to consider.

  Case 1 *job n is not in $\mathcal{O}$.*

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    - *$\mathcal{O}$ cannot use incompatible jobs $\{p(n)+1, p(n)+2, \ldots, n-1\}$.*
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- $\mathcal{O}$ must be the best of these two choices!
- Suggests finding optimal solution for sub-problems consisting of jobs $\{1, 2, \ldots, j-1, j\}$, for all values of j.

Recursion

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- When does request j belong to $\mathcal{O}_j$? If and only if $v_j + \text{OPT}(p(j)) \geq \text{OPT}(j-1)$.

Recursive Algorithm

```
Algorithm: Compute-opt
    input  : A set of weighted jobs $R$, index $j$ and largest
             compatible indices.
    output: A set of compatible jobs $A$
if $j = 0$ then
    return 0
else
    return max
      ($v_j$+Compute−opt(p(j)),Compute−opt(j-1))
end
```

- Correctness of algorithm follows by induction.
- What is the running time of the algorithm?

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- What is the running time of the algorithm? Can be exponential in n.
- When $p(j) = j - 2$, for all $j \geq 2$: recursive calls are for $j - 1$ and $j - 2$.

Example of Recursive Algorithm

$\text{OPT}(6) =$

$\text{OPT}(5) =$

$\text{OPT}(4) =$

$\text{OPT}(3) =$

$\text{OPT}(2) =$

$\text{OPT}(1) =$

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Example of Recursive Algorithm

$$\begin{aligned}
\mathrm{OPT}(6) &= \max(v_6 + \mathrm{OPT}(p(6)), \mathrm{OPT}(5)) = \max(1 + \mathrm{OPT}(3), \mathrm{OPT}(5))\\
\mathrm{OPT}(5) &=\\
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- Optimal solution is

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- Optimal solution is job 5, job 3, and job 1.

Memoisation

- Store OPT(j) values in a cache and reuse them rather than recompute them.

```
Algorithm: M-Compute-opt
input  : A set of weighted jobs $R$, index $j$ and largest
         compatible indices.
output: A set of compatible jobs $A$

if $j = 0$ then
    return 0;
else if $M[j]$ is not empty then
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    $M[j]$= max
        ($v_j$+M-Compute−opt(p(j)),M-Compute−opt(j-1));
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Running Time of Memoisation

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- Total time spent is the order of the number of recursive calls to `M-Compute-Opt`.

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- Each time `M-Compute-Opt` issues two recursive calls, it fills in a new entry in M.
- Therefore, total number of recursive calls is $\boxed{O(n)}$.

From Recursion to Iteration

- Unwind the recursion and convert it into iteration.
- Can compute values in M iteratively in $O(n)$ time.
- `Find-Solution` works as before.

Algorithm: Iterative weighted interval scheduling

input : A set of weighted jobs R, index j and largest compatible indices.

output: A set of compatible jobs A

```
M[0] = 0;
foreach $j \in [1, n]$ do
    M[j] = max($v_j$+M[p(j)],M[j-1]);
end
```

Basic Outline of Dynamic Programming

- To solve a problem, we need a collection of sub-problems that satisfy a few properties:
  1. There are a polynomial number of sub-problems.
  2. The solution to the problem can be computed easily from the solutions to the sub-problems.
  3. There is a natural ordering of the sub-problems from "smallest" to "largest".
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- Difficulties in designing dynamic programming algorithms:
  1. Which sub-problems to define?
  2. How can we tie up sub-problems using a recurrence?
  3. How do we order the sub-problems (to allow iterative computation of optimal solutions to sub-problems)?

Algorithm design and analysis

— Some exercises —

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Pontifical Catholic University of Minas Gerais – PUC Minas

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Maximum subarray problem

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Some properties of this problem are:

- If the array contains all non-negative numbers, then the problem is trivial
- If the array contains all non-positive numbers, then a solution is any subarray of size 1;
- Several different sub-arrays may have the same maximum sum.

Maximum subarray problem

A>0

x

B>0

Maximum subarray problem

- If $x > 0$ then the answer is A + x + B
- If $x < 0$ then the answer may be
  1. $\max\{A, B\}$ if $A + x < 0$
  2. $\max\{A, B, A + x + B\}$ if $A + x > 0$

Maximum subarray problem

Longest increasing subsequence

The **longest increasing subsequence** problem is to find a subsequence of a given sequence in which the subsequence's elements are in **sorted order**, lowest to highest, and in which the subsequence is as long as possible.

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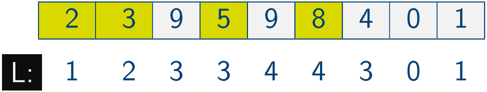

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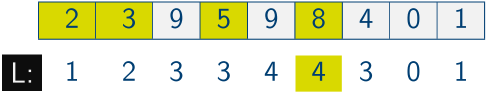

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- There are n possible places where you can place an advertisement given by $x_1, x_2, \cdots, x_n$ in $[0, M]$.
- Placing an advertisement at x_i gives value r_i.
- You cannot put two advertisements at distance < 5kms from each other.

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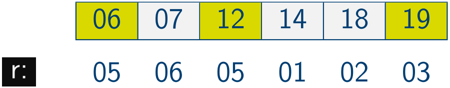

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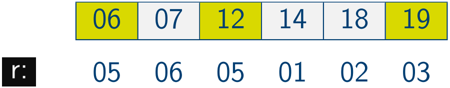

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Placing billboards

Algorithm design and analysis

— Segmented Least Squares —

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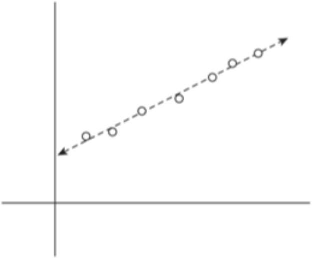
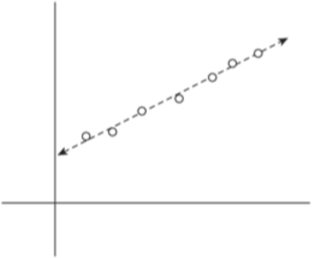
Least Squares Problem



- Given scientific or statistical data plotted on two axes.
- Find the “best” line that “passes” through these points.

Least Squares Problem



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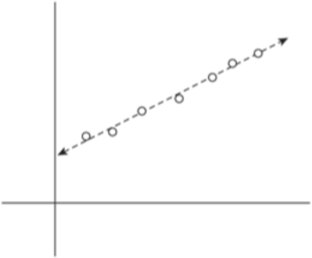

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LEAST SQUARES

INSTANCE Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of n points.

SOLUTION Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

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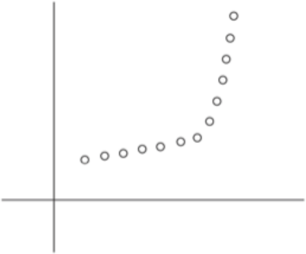
- Solution is achieved by

$$a = \frac{n \sum_i x_i y_i - \left(\sum_i x_i\right)\left(\sum_i y_i\right)}{n \sum_i x_i^2 - \left(\sum_i x_i\right)^2} \text{ and } b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares



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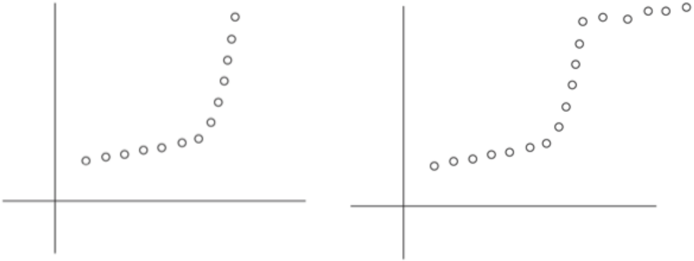
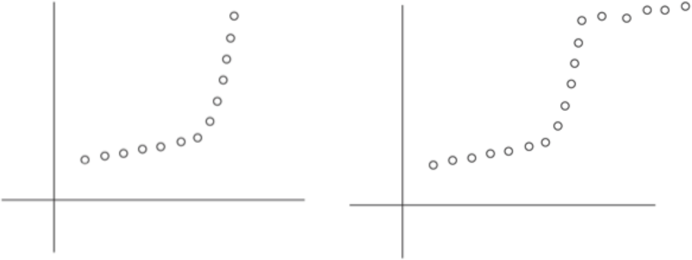
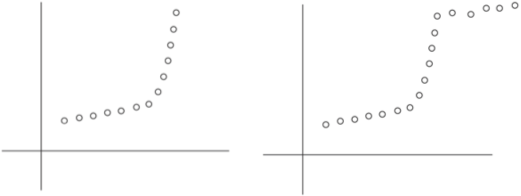


Segmented Least Squares



- Want to fit multiple lines through P.
- Each line must fit contiguous set of x-coordinates.
- Lines must minimise total error.

Segmented Least Squares



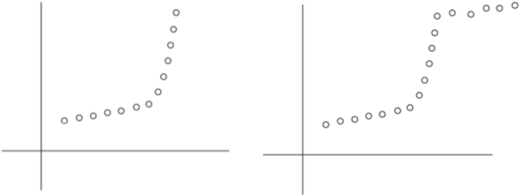
SEGMENTED LEAST SQUARES

INSTANCE Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of n points, $x_1 < x_2 < \cdots < x_n$.

SOLUTION A integer k, a partition of P into k segments $\{P_1, P_2, \ldots, P_k\}$, k lines $L_j : y = a_j x + b_j, 1 \leq j \leq k$ that minimise

$$\sum_{j=1}^{k} \text{Error}(L_j, P_j)$$

Segmented Least Squares



SEGMENTED LEAST SQUARES

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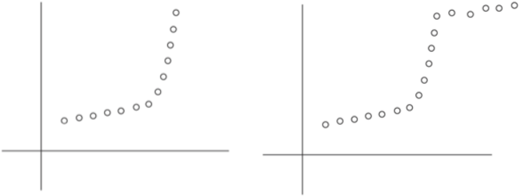

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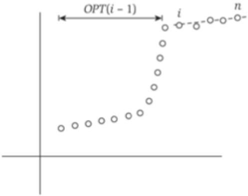
A subset P' of P is a **segment** if $1 \leq i < j \leq n$ exist such that $P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\}$.

Formulating the Recursion: I

- Observation: p_n is part of some segment in the optimal solution. This segment starts at some point p_i.
- Let $\mathrm{OPT}(i)$ be the optimal value for the points $\{p_1, p_2, \ldots, p_i\}$.
- Let $e_{i,j}$ denote the minimum error of any line that fits $\{p_i, p_2, \ldots, p_j\}$.
- We want to compute $\mathrm{OPT}(n)$.



- If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

$$\mathrm{OPT}(n) = e_{i,n} + C + \mathrm{OPT}(i-1)$$

Formulating the Recursion: II

- Consider the sub-problem on the points $\{p_1, p_2, \ldots p_j\}$
- To obtain $\mathrm{OPT}(j)$, if the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_j\}$, then

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$$\mathrm{OPT}(j) = e_{i,j} + C + \mathrm{OPT}(i-1)$$

- Since i can take only j distinct values,

$$\mathrm{OPT}(j) = \min_{1 \leq i \leq j} \left(e_{i,j} + C + \mathrm{OPT}(i-1)\right)$$

- Segment $\{p_i, p_{i+1}, \ldots p_j\}$ is part of the optimal solution for this sub-problem if and only if the minimum value of $\mathrm{OPT}(j)$ is obtained using index i. solution

Dynamic Programming Algorithm

$$OPT(j) = \begin{cases} 0, & \text{if } j = 0 \\ \min_{1 \le i \le j}(e_{ij} + c + OPT[i-1]), & \text{otherwise} \end{cases}$$

Algorithm: Segmented least squares: an iterative algorithm

```
input  : A set of $n$ points $p_i$
output: A set of compatible jobs $A$
M[0] = 0;
for $j$=1 to $n$ do
    for $i$=1 to $j$ do
        compute the $e_{ij}$ for the segment $p_i, \cdots, p_j$;
    end
end
for $j$=1 to $n$ do
    M[j] = $\min_{1 \le i \le j}(e_{ij} + c + M[i-1])$;
end
```

Dynamic Programming Algorithm

$$OPT(j) = \begin{cases} 0, & \text{if } j = 0 \\ \min_{1 \le i \le j}(e_{ij} + c + OPT[i-1]), & \text{otherwise} \end{cases}$$

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```

- Running time is $O(n^3)$, can be improved to $O(n^2)$.
- We can find the segments in the optimal solution by backtracking.

Algorithm design and analysis

— Sequence alignment —

Silvio Guimarães

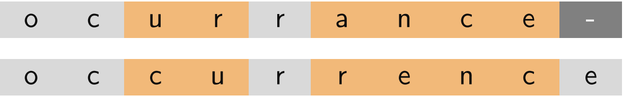
Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2023

Sequence Similarity

- Given two strings, measure how similar they are.
- Given a database of strings and a query string, compute the string most similar to query in the database.
- Applications:
    - Online searches (Web, dictionary).
    - Spell-checkers.
    - Computational biology
    - Speech recognition.
    - Basis for Unix `diff`.

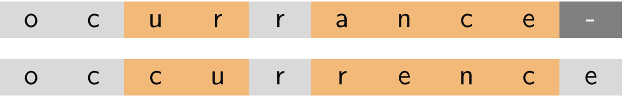
Defining Sequence Similarity



o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

Defining Sequence Similarity



o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

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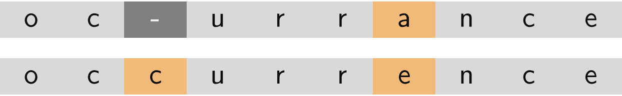

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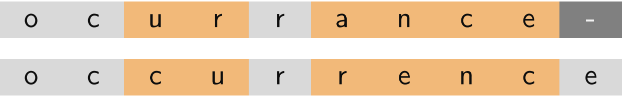

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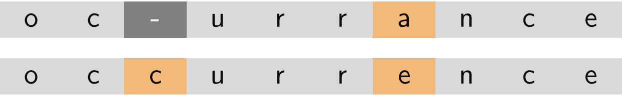


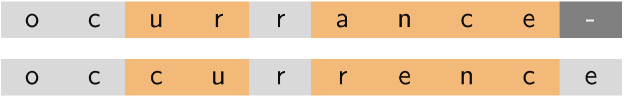
o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

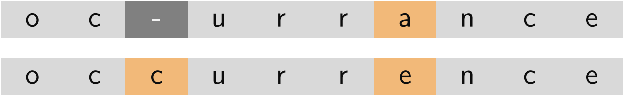
0 mismatches, 3 gaps

Defining Sequence Similarity



```
o c u r r a n c e -
o c c u r r e n c e
```

6 mismatches, 1 gap



```
o c - u r r a n c e
o c c u r r e n c e
```

1 mismatch, 1 gap

```
o c - u r r - a n c e
o c c u r r e - n c e
```

0 mismatches, 3 gaps

- Edit distance model: how many changes must you to make to one string to transform it into another?
- Changes allowed are deleting a letter, adding a letter, changing a letter.

Edit Distance

EDIT DISTANCE

INSTANCE Let two string $x = x_1x_2x_3 \dots x_m$ and $y = y_1y_2 \dots y_n$

SOLUTION An alignment of minimum cost.

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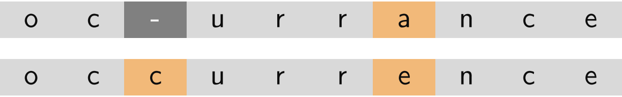

o	c	-	u	r	r	a	n	c	e
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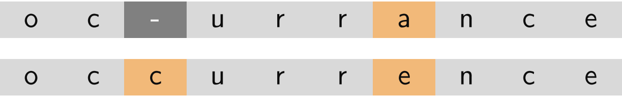

o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

- A **matching** of these sets is a set M of ordered pairs such that
  1. in each pair (i, j), $1 \leq i \leq m$ and $1 \leq j \leq m$ and
  2. no index from x (respectively, from y) appears as the first (respectively, second) element in more than one ordered pair.
- A matching M is an **alignment** if there are no “crossing pairs” in M: if $(i, j) \in M$ and $(i', j') \in M$ and $i < i'$ then $j < j'$.

Edit Distance

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INSTANCE Let two string $x = x_1 x_2 x_3 \ldots x_m$ and $y = y_1 y_2 \ldots y_n$

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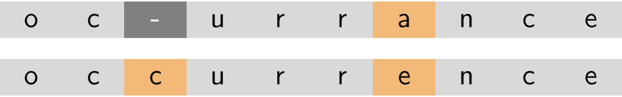

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o	c	c	u	r	r	e	n	c	e

- A matching M is an alignment if there are no “crossing pairs” in M: if $(i, j) \in M$ and $(i', j') \in M$ and $i < i'$ then $j < j'$.
- The pair x_i-y_j and $x_{i'}$-yj' cross if $i < i'$’, but $j - j'$.

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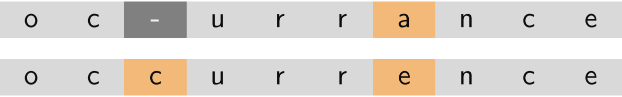

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- A matching M is an alignment if there are no "crossing pairs" in M: if $(i,j) \in M$ and $(i',j') \in M$ and $i < i'$ then $j < j'$.

$$cost(M) = \underbrace{\sum_{(x_i,y_j)\in M} \alpha_{x_iy_j}}_{\text{mismatch}} + \underbrace{\sum_{i:x_i\ \ unmatched} \delta + \sum_{j:x_j\ \ unmatched} \delta}_{\text{gaps}}$$

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. **Is $(m, n) \in M$?**
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.

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- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.
- $\text{OPT}(i, j)$ = min cost of aligning $x = x_1 \dots x_i$ and $y = y_1 \dots y_j$.
    - Case 1: OPT matches x_i-y_j so $(i, j) \in M$:

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$$\mathrm{OPT}(i, j) = \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1)$$

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.

  - Case 2a: OPT leaves x_i unmatched, so i not matched:

$$\mathsf{OPT}(i, j) = \delta + \mathsf{OPT}(i-1, j)$$

.

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
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    $$\mathrm{OPT}(i, j) = \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1)$$

    .
  - Case 2a: OPT leaves x_i unmatched, so i not matched:

    $$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i-1, j)$$

    .
  - Case 2b: OPT leaves y_j unmatched, so j not matched:

    $$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i, j-1)$$

    .

Dynamic Programming Approach

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    .
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    .
  - $(i, j) \in M$ if and only if minimum is achieved by the first term.
- What are the base cases?

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
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    .
  - $(i, j) \in M$ if and only if minimum is achieved by the first term.
- What are the base cases? $\mathrm{OPT}(i, 0) = \mathrm{OPT}(0, i) = i\delta$.

Dynamic Programming Algorithm

$$\mathrm{OPT}(i,j) = \begin{cases} j\delta, & \textit{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1), \\ \delta + \mathrm{OPT}(i-1, j), \\ \delta + \mathrm{OPT}(i, j-1) \end{cases} & \textit{otherwise} \\ i\delta, & \textit{if } j = 0 \end{cases}$$

- Running time is $O(mn)$. Space used in $O(mn)$.

Dynamic Programming Algorithm

$$\mathrm{OPT}(i,j) = \begin{cases} j\delta, & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1), \\ \delta + \mathrm{OPT}(i-1, j), \\ \delta + \mathrm{OPT}(i, j-1) \end{cases} & \text{otherwise} \\ i\delta, & \text{if } j = 0 \end{cases}$$

- Running time is $O(mn)$. Space used in $O(mn)$.
- Can compute $\mathrm{OPT}(m,n)$ in $O(mn)$ time and $O(m+n)$ space (*Hirschberg 1975*, Chapter 6.7).

Dynamic Programming Algorithm

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- Running time is $O(mn)$. Space used in $O(mn)$.
- Can compute $\mathrm{OPT}(m,n)$ in $O(mn)$ time and $O(m+n)$ space (*Hirschberg 1975*, Chapter 6.7).
- Can compute *alignment* in the same bounds by combining dynamic programming with divide and conquer.

Dynamic Programming Algorithm

Longest commom subsequence

The **longest commom subsequence** problem is the task of finding the longest subsequence which is in two sequences x and y.

Longest commom subsequence

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Formally, $w_0 w_1 \ldots w_{i-1}$ is a subsequence of $x_0 x_1 \ldots x_{m-1}$ if there exists a **strictly increasing** sequence of integers $(k_0, k_1, \ldots, k_{i-1})$ such that for $0 \leq k \leq i-1$. A word w is a longest common subsequence of x and y if w is a subsequence of x, a subsequence of y and its length is maximal.

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C	T	A	C	C	G	A	
T	A	C	A	T	T	G	T

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C	T	A	C	C	G	A	
T	A	C	A	T	T	G	T

Some properties of this problem are:

- the lenght of the longest subsequence must be maximal;
- may have several longest subsequences with the same size;
- it is possible to identify the subsequence by backtracking

Dynamic Programming Algorithm

$$\text{OPT}(i,j) = \begin{cases} 0, & \textit{if } i = 0 \\ 1 + \text{OPT}(i-1, j-1), & \textit{if } x_i = y_j \\ \max \begin{cases} \text{OPT}(i-1, j), \\ \text{OPT}(i, j-1) \end{cases} & \textit{otherwise} \\ 0, & \textit{if } j = 0 \end{cases}$$

	C	T	A	C	C
T					
A					
C					
A					
C					
G					

Dynamic Programming Algorithm

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		C	T	A	C	C
	0	0	0	0	0	0
T						
A						
C						
A						
C						
G						

Dynamic Programming Algorithm

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		C	T	A	C	C
	0	0	0	0	0	0
T	0	0	1	1	1	1
A						
C						
A						
C						
G						

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		C	T	A	C	C
	0	0	0	0	0	0
T	0	0	1	1	1	1
A	0	0	1	2	2	2
C						
A						
C						
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Dynamic Programming Algorithm

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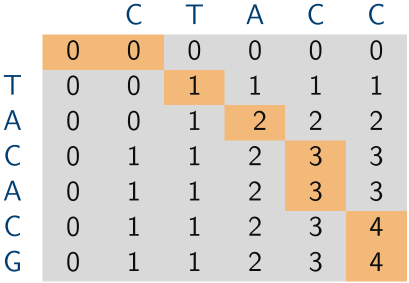

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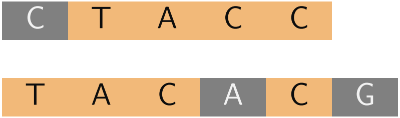

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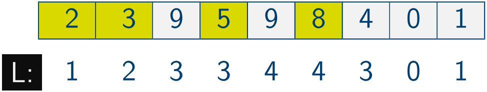

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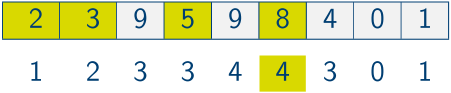

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Algorithm design and analysis

— Shortest Path Problem —

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2023

Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
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SHORTEST PATHS

INSTANCE A directed graph $G(V, E)$, a function $l : E \to \mathbb{R}^+$, and a node $s \in V$

SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

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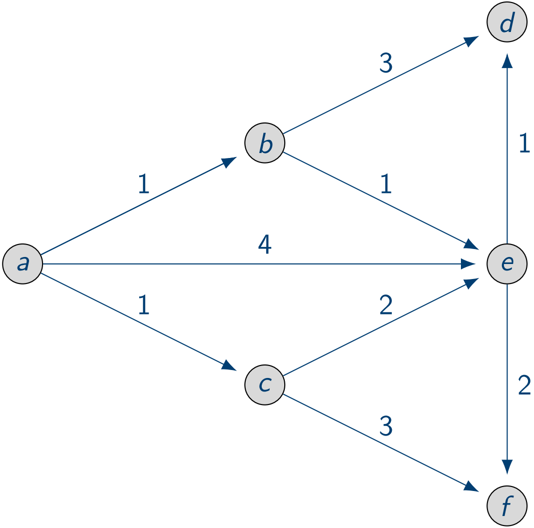
- Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimises $d'(v)$.

Example of Dijkstra's Algorithm



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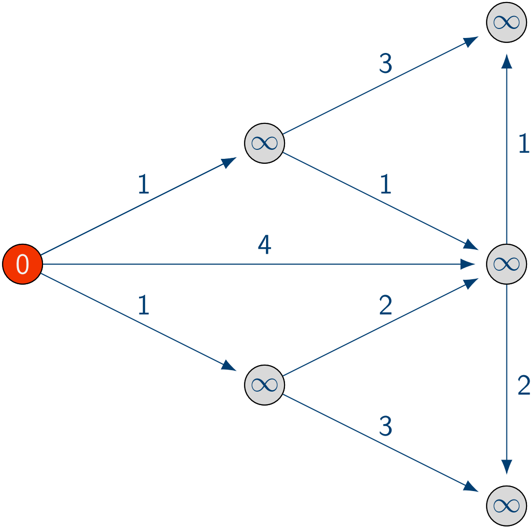

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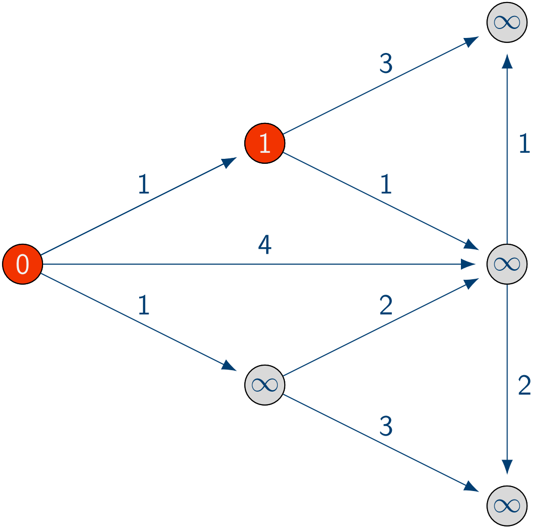

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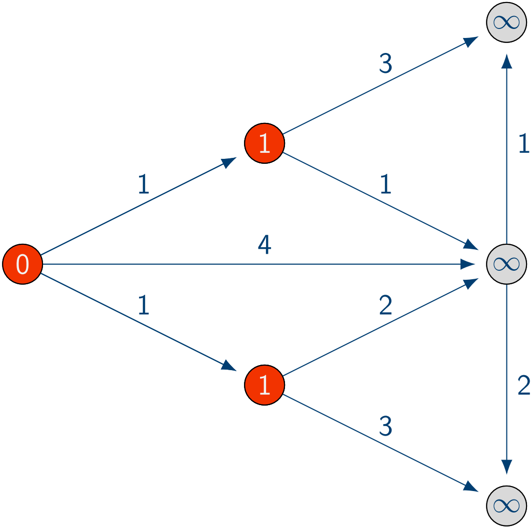

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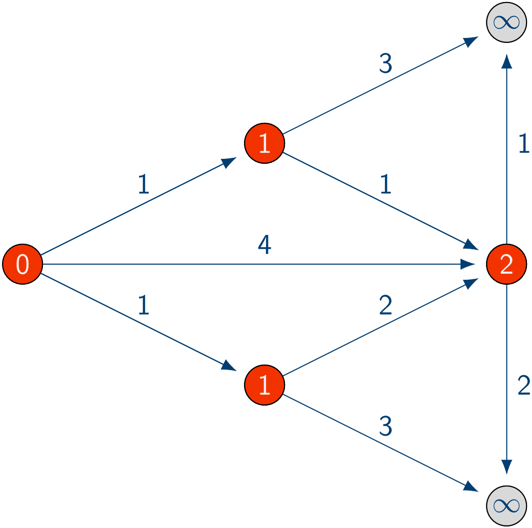

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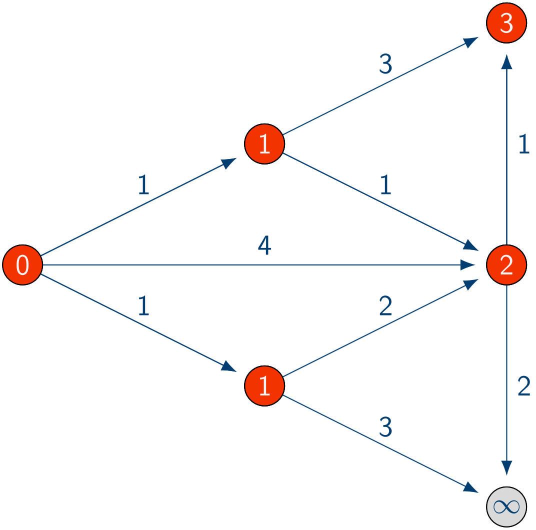

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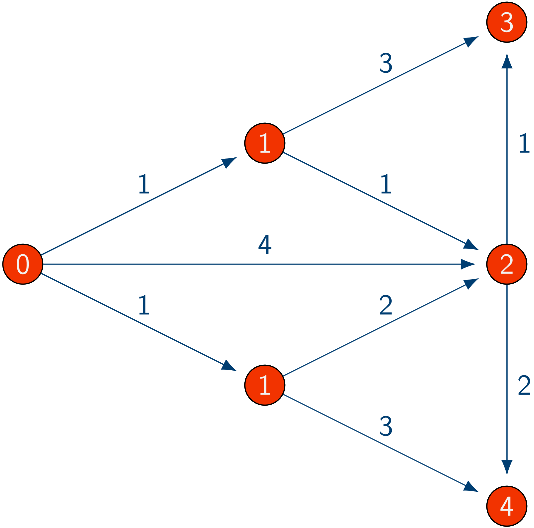

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.
  - Base case: $|S| = 1$. The only node in S is s.
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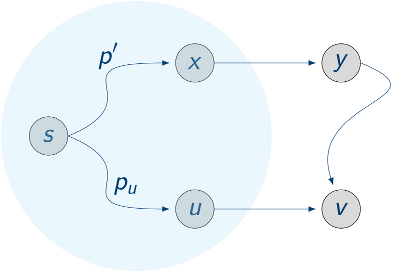

The alternate $s - v$ path P through x and y already too long by the time it had left the set S

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

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- In each iteration, for each node $v \notin S$, compute

$$\min_{e=(u,v),u \in S} d(u) + l_e.$$

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
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- $G = (V, E)$ is a connected directed graph. Each edge e has a length l_e. Note that the weights may be negative.
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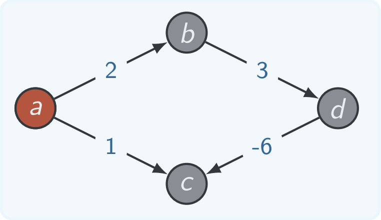
INSTANCE	A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}$, and a node $s \in V$
SOLUTION	A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Bellman-Ford Algorithm

Dijkstra – Can fail if negative edge costs.



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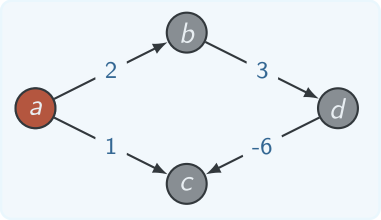
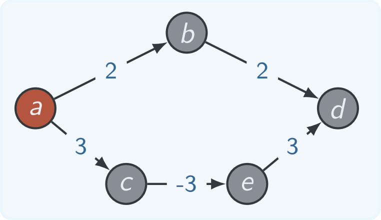


Re-weighting – Adding a **constant** to every edge weight can fail



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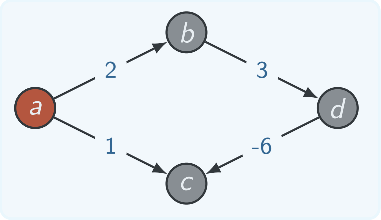

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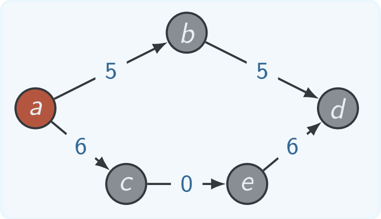
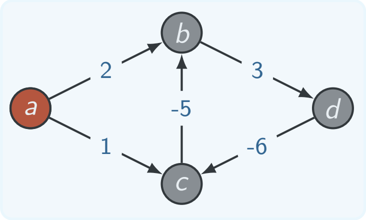


Bellman-Ford Algorithm

If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; otherwise, there exists one that is simple.



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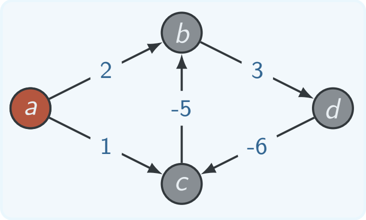

The Bellman-Ford algorithm is a way to find single source shortest paths in a graph with negative edge weights (but no negative cycles).

Bellman-Ford Algorithm

OPT(i, v) = length of shortest v-t path P using at most i edges.

Bellman-Ford Algorithm

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- Case 2: *P* uses exactly *i* edges
  - if (v, w) is first edge, then OPT uses (v, w), and then selects best *w*-*t* path using at most $i-1$ edges

Bellman-Ford Algorithm

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$$OPT(i,v) = \begin{cases} 0, & if \;\; i = 0 \\ \min \begin{cases} OPT(i-1,v) \\ \min\{OPT(i-1,w) + c_{vw}\} \end{cases}, & otherwise \end{cases}$$

A Faster implementation of Dijkstra's Algorithm

```
Algorithm: Shortest path algorithm – Bellman-Ford
   input  : A graph $G = (V, E)$, a weight map $W$ and a source node $s$.
   output: The distances of the vertices from $s$

foreach $v \in V$ do $d[0, u] = \infty$;
Initially $d[0, s] = 0$;

for $i = 1$ to $n - 1$ do
    foreach $v \in V$ do
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    foreach edge $(v, w) \in E$ do
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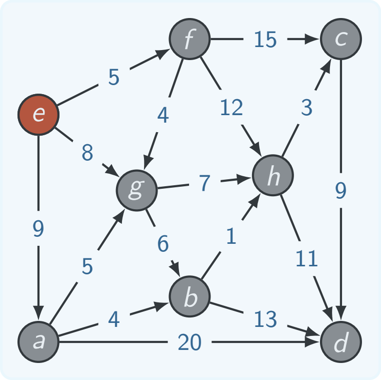
How to detect negative cycles?

Shortest path – an example



Compute the shortest path from *e* to all other nodes!