Algorithm design and analysis

— Network Flow —

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Image and Multimedia Data Science Laboratory – IMScience
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— Maximum Flow and Minimum Cut —

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Maximum Flow and Minimum Cut

- Two rich algorithmic problems.
- Fundamental problems in combinatorial optimization.
- Beautiful mathematical duality between flows and cuts.
- Numerous non-trivial applications:
  - Bipartite matching.
  - Data mining.
  - Project selection.
  - Airline scheduling.
  - Baseball elimination.
  - Image segmentation.
  - Network connectivity.
  - Open-pit mining.
  - Network reliability.
  - Distributed computing.
  - Egalitarian stable matching.
  - Security of statistical data.
  - Network intrusion detection.
  - Multi-camera scene reconstruction.
  - Gene function prediction.

Flow Networks

- Use directed graphs to model transportation networks:
  - edges carry traffic and have capacities.
  - nodes act as switches.
  - *source* nodes generate traffic, *sink* nodes absorb traffic.

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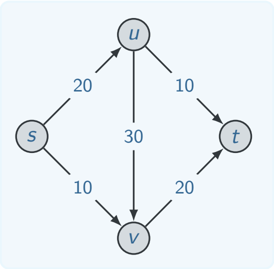

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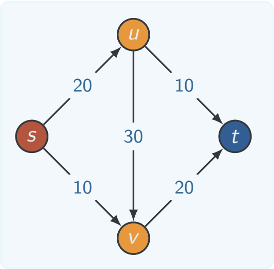

- A flow network is a directed graph $G = (V, E)$
  - Each edge $e \in E$ has a capacity $c(e) > 0$.
  - There is a single source node $s \in V$.
  - There is a single sink node $t \in V$.
  - Nodes other than s and t are internal.

Defining Flow

- In a flow network $G = (V, E)$, an s-t flow is a function $f : E \to \mathbb{R}^+$ such that
  - (i) Capacity conditions For each $e \in E$, $0 \leq f(e) \leq c(e)$.
  - (ii) Conservation conditions For each internal node v,

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

- The value of a flow is $\nu(f) = \sum_{e \text{ out of } s} f(e)$.

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- The value of a flow is $\nu(f) = \sum_{e \text{ out of } s} f(e)$.
- Useful notation:

$f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$ $\quad$ $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$

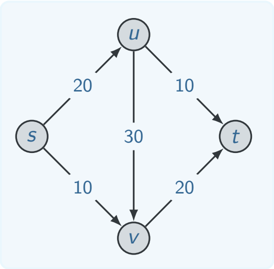
For $S \subseteq V$,

$f^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)$ $\quad$ $f^{\text{in}}(S) = \sum_{e \text{ into } S} f(e)$

Maximum Flow

INSTANCE A flow network G

SOLUTION The flow with largest value in G



- Assumptions:
  1. No edges enter s, no edges leave t.

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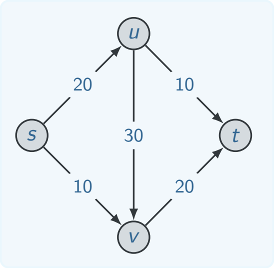

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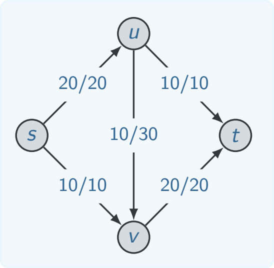

- Assumptions:
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  3. All edge capacities are integers.

Algorithm design and analysis

— Ford-Fulkerson Algorithm —

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Developing the Algorithm

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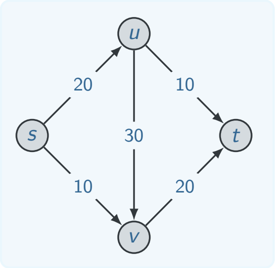

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- Let us take a greedy approach.
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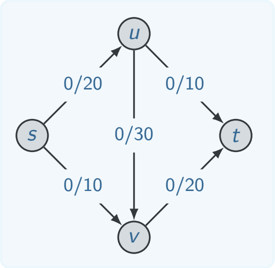

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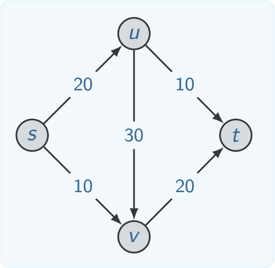

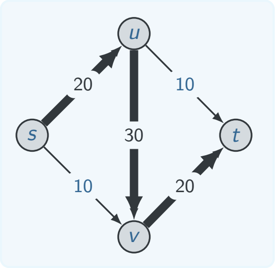

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- Let us take a greedy approach.
  1. Start with zero flow along all edges.
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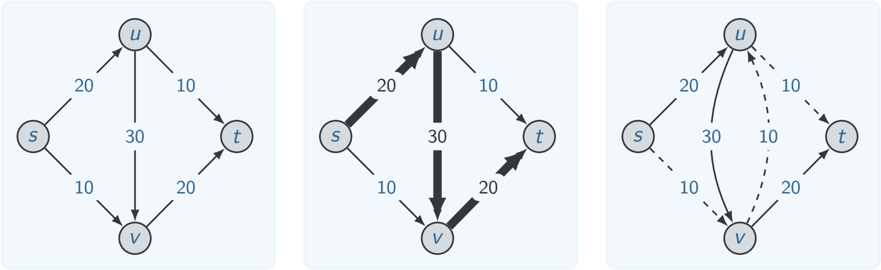
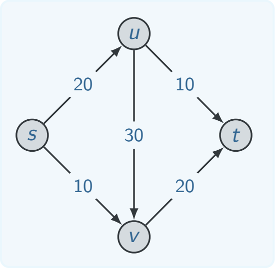
  3. Key idea: Push flow along edges with leftover capacity and undo flow on edges already carrying flow.



Residual Graph

- Given a flow network $G = (V, E)$ and a flow f on G, the residual graph G_f of G with respect to f is a directed graph such that
  (i) *Nodes* – G_f has the same nodes as G.



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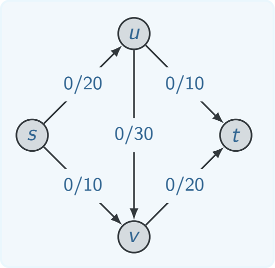

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  - (ii) **Forward** edges – For each edge $e = (u, v) \in E$ such that $f(e) < c(e)$, G_f contains the edge (u, v) with a **residual capacity** $c(e) - f(e)$.
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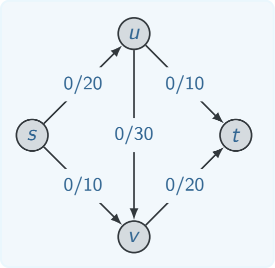

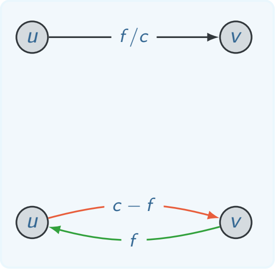

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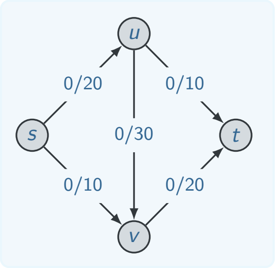
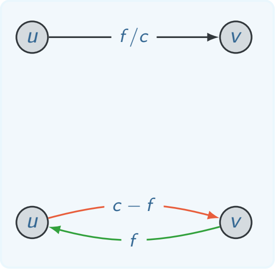
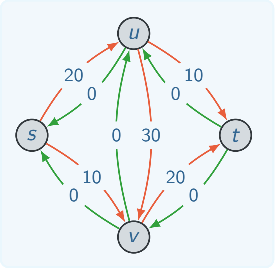
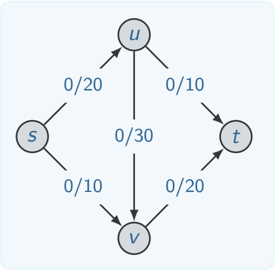






Augmenting Paths in a Residual Graph

- Let P be a simple s-t path in G_f.
- bottleneck(P, f) is the minimum residual capacity of any edge in P.
- The following operation `augment`(f, P) yields a new flow f' in G:



Algorithm: Augmented path

input : A graph $G = (V, E)$, a path P and a source s and a sink t nodes.
output: The distances of the vertices from s

```
Let b = bottleneck(P, f) ;
foreach edge e = (u, v) ∈ P do
    if e is a forward edge then
        increase f(e) in G by b
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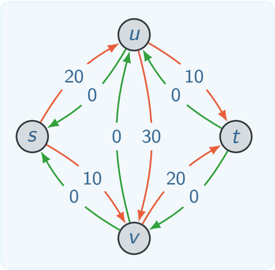

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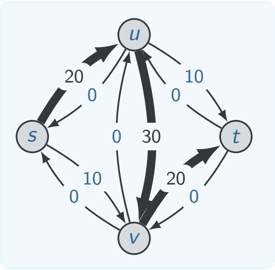

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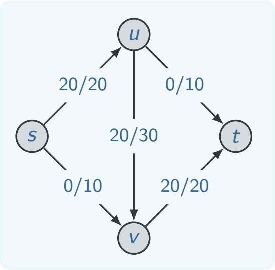

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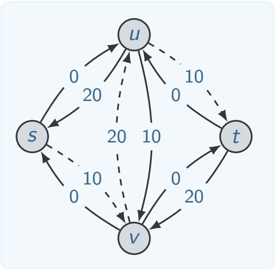

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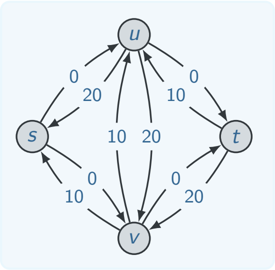

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- A simple s-t path in the residual graph is an augmenting path.
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  - Conservation condition on internal node $v \in P$. Four cases to work out.

Ford-Fulkerson Algorithm

Algorithm: Ford-Fulkerson Algorithm

input : A graph $G = (V, E)$, a source s and a sink t nodes.

output: The flow f

$f(e) = 0, \forall e \in E$;
while *there is a path s-t in the residual graph* G_f **do**
  Let P be a simple s-t path in G_f;
  $f' = augment(f, P)$;
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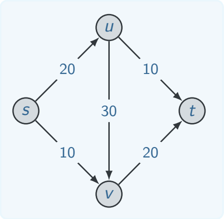

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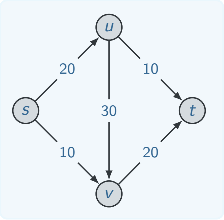

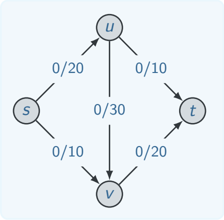

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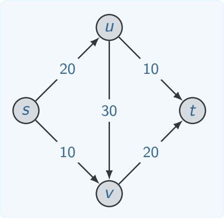

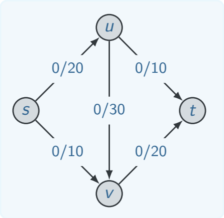

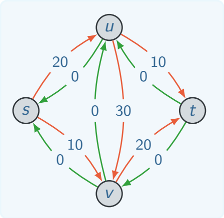

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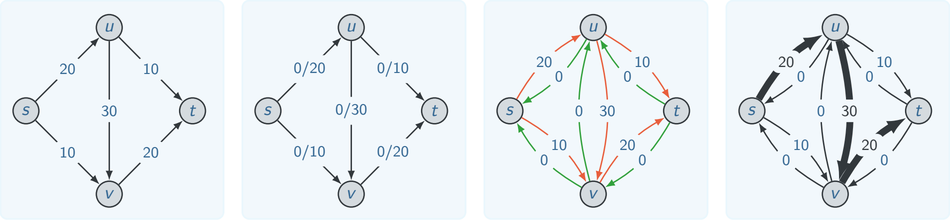

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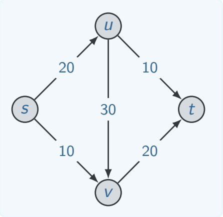

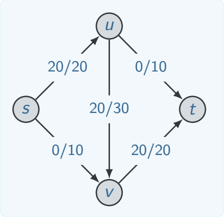

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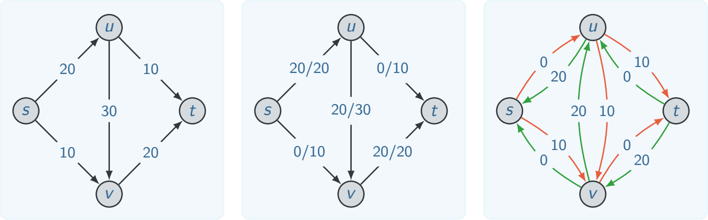

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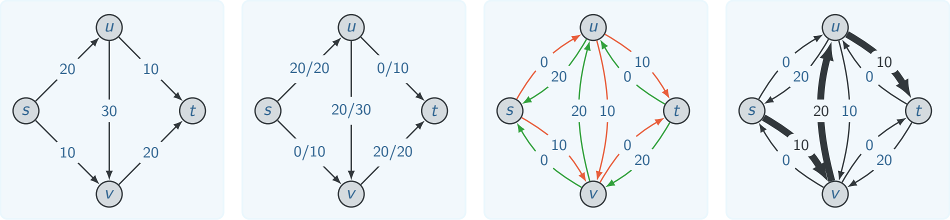

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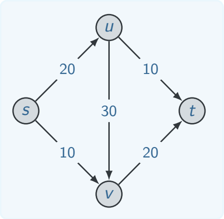

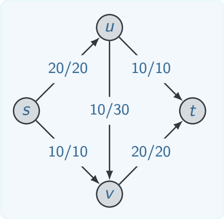

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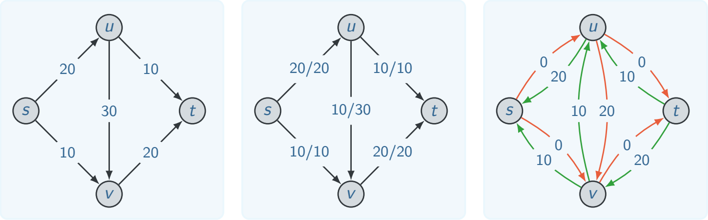

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- Is there a better bound?
- Idea: An **s-t cut** is a partition of V into sets A and B such that $s \in A$ and $t \in B$.
  - **Capacity** of the cut (A, B) is $c(A, B) = \sum_{e \text{ out of } A} c(e)$.
  - Intuition: For every flow f, $\nu(f) \leq c(A, B)$.

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- Question: Is the reverse true? Is the smallest capacity of a cut at most the maximum flow?
- Answer: Yes, and the Ford-Fulkerson algorithm computes this cut!

Flows and Cuts

- Let $\bar{f}$ denote the flow computed by the Ford-Fulkerson algorithm.
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- Claim: If f is an s-t flow such that G_f has no s-t path, then there is an s-t cut (A^*, B^*) such that $\nu(f) = c(A^*, B^*)$.
  - Claim applies to *any* flow f such that G_f has no s-t path, and not just to the flow $\bar{f}$ computed by the Ford-Fulkerson algorithm.

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- Claim: f is an s-t flow and G_f has no s-t path $\Rightarrow \exists$ s-t cut (A^*, B^*), $\nu(f) = c(A^*, B^*)$.
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- In every flow network, there is a flow f and a cut (A, B) such that $\nu(f) = c(A, B)$.
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- Corollary: If all capacities in a flow network are integers, then there is a maximum flow f where every flow value $f(e)$ is an integer.

Algorithm design and analysis

— Scaling Max-Flow Algorithm —

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
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Feb 2023

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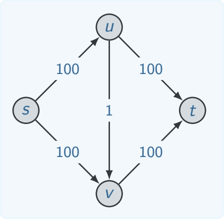

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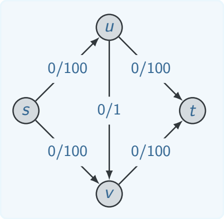

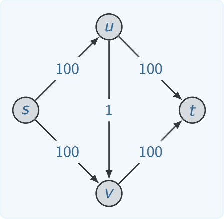

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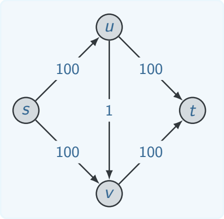

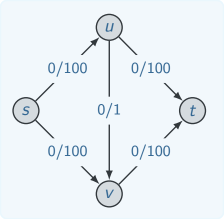

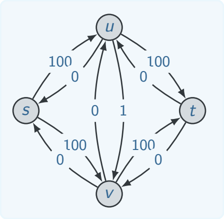

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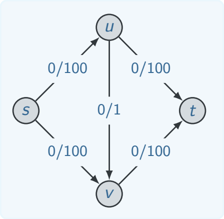

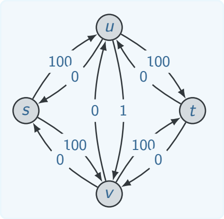

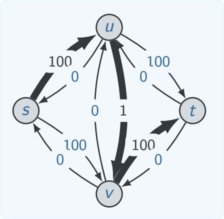

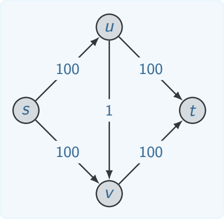

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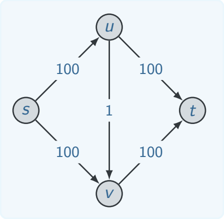

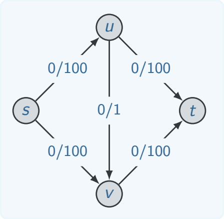

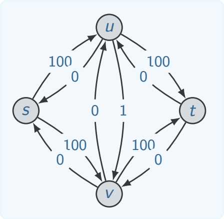

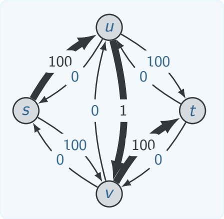

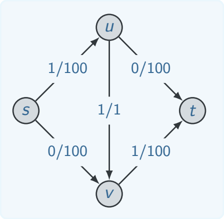

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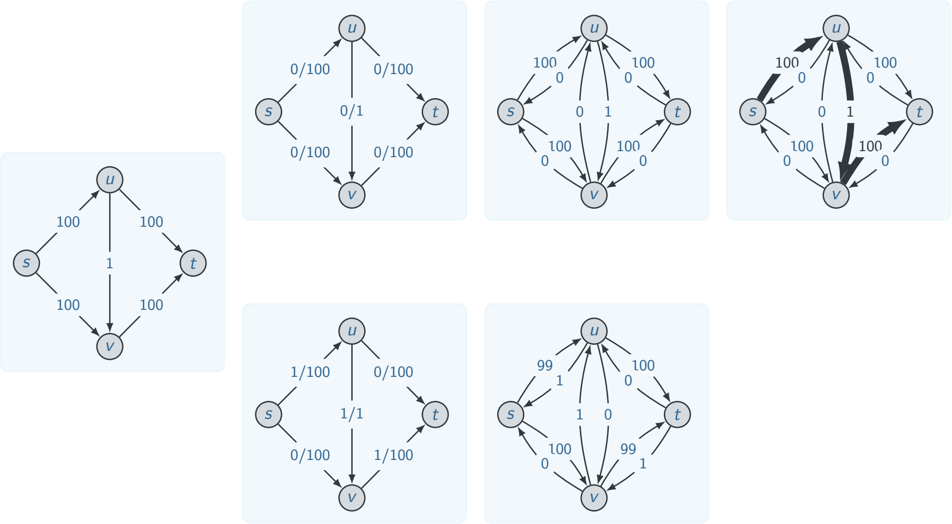

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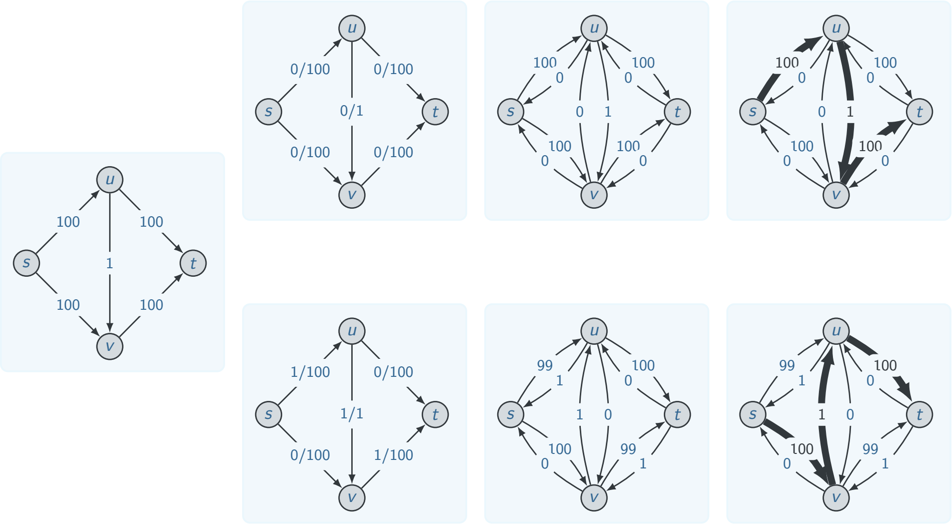

Improving Ford-Fulkerson Algorithm

- Bad case for Ford-Fulkerson algorithm is when the bottleneck edge is the augmenting path has a low capacity.
- Idea: decrease number of iterations by picking s-t path with bottleneck edge of largest capacity.

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- Idea: decrease number of iterations by picking s-t path with bottleneck edge of largest capacity. Computing this path can slow down each iteration considerably.

Other Maximum Flow Algorithms

- Running time of the Ford-Fulkerson algorithm is $O(mC)$, which is pseudo-polynomial: polynomial in the magnitudes of the numbers in the input.
- Desire a strongly polynomial algorithm: running time is depends only on the *size* of the graph and is *independent* of the numerical values of the capacities (as long as numerical operations take $O(1)$ time).

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- Edmonds-Karp, Dinitz: choose augmenting path to be the shortest path in G_f (use breadth-first search). Algorithm runs in $O(mn)$ iterations.
- Improved algorithms take time $O(mn \log n)$, $O(n^3)$, etc. on augmenting paths. Runs in $O(n^2 m)$ or $O(n^3)$ time.

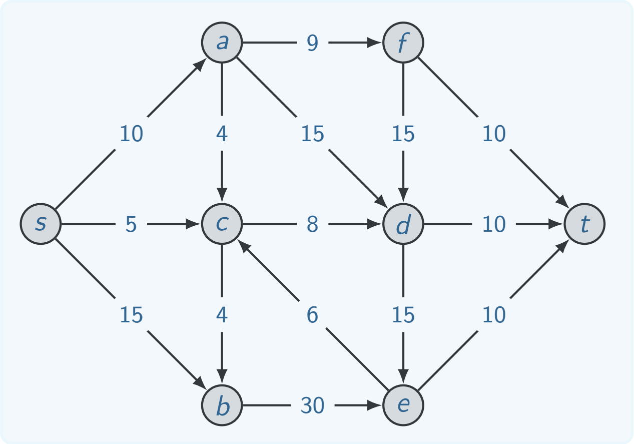
Algorithm design and analysis

— Exercises —

Silvio Guimarães

Graduate Program in Informatics – PPGINF
Image and Multimedia Data Science Laboratory – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Compute the maximum flow



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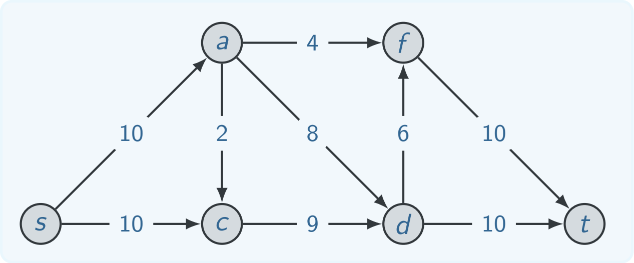
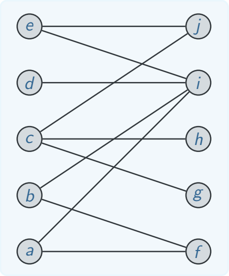


Bipartite graph matching

BIPARTITE GRAPH MATCHING

INSTANCE Let $G = (L \cup R, E)$ be an undirected graph. $M \subseteq E$ is a matching if each node appear in, at most, one edge in M.

SOLUTION Find a max cardinality matching.



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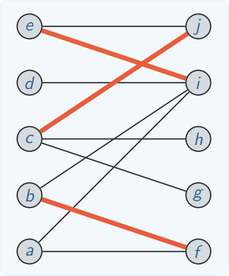

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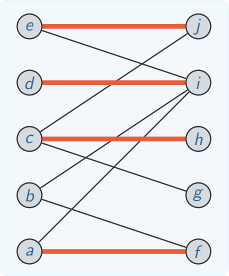

Edge Disjoint Paths

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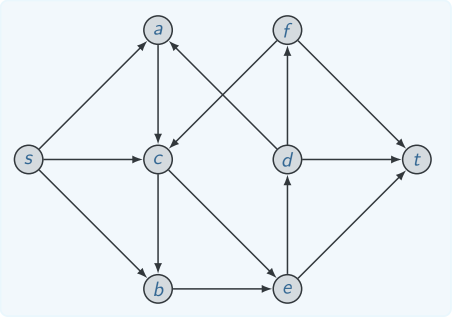

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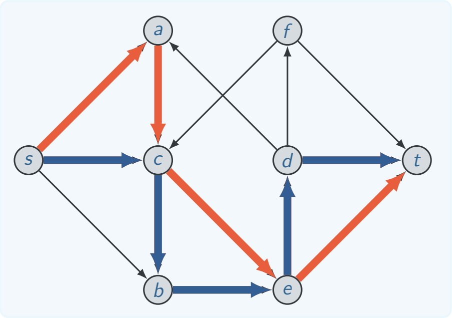

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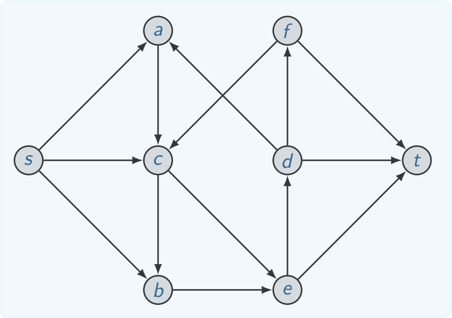

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